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Small Angle Scattering of 970 MeV Protons from Carbon (*).

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1. - Introduction.

The differential cross-section for the scattering at small angles of 970 MeV protons from Carbon has been measured in order to examine the interference between the Coulomb and nuclear parts of the scattering at this energy. The measurements have been made at laboratory angles from 1° to $3\frac{3}{4}^\circ$.

2. - Experimental arrangement.

The proton beam of the Birmingham synchrotron was scattered from a carbon target placed inside the vacuum box. Protons elastically scattered from this target through $5\frac{1}{2}^\circ$, passed through a thin window, and were focussed to a nearly parallel beam by a set of steel focussing shims [1] attached to the synchrotron magnet. This beam then passed through a hole in the concrete shielding wall into the experimental area. The vertical height of the beam was reduced by use of a tungsten alloy channel; a carefully positioned iron channel then allowed the main beam to pass through it but removed unwanted background. (See Fig. 1).

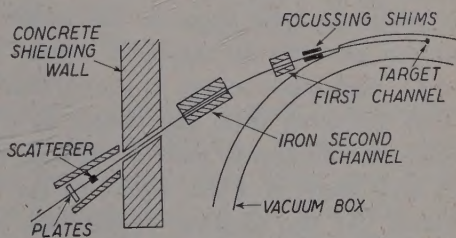


Fig. 1. - Experimental arrangement (Not to scale).

At the experimental position the beam was 4 cm high and had a horizontal

(*) Lavoro presentato al XLIII Congresso Nazionale della Società Italiana di Fisica, Padova-Venezia, 22-28 Settembre 1957.

distribution which was close to that of a gaussian with standard deviation 1 cm. The total beam intensity was $\sim 10^4$ particles/pulse. The scattering target for the experiment was placed at a point 2 metres from the exit of the iron channel and particles scattered by the target were observed in photographic emulsions placed 1 metre away from it. The unscattered beam also passed through the centre of the plates, giving an angle to which the scatterings could be referred.

3. - Observations.

At any point on one of the plates tracks of protons scattered from the carbon block, together with a usually much larger number emerging from the iron channel, can be found.

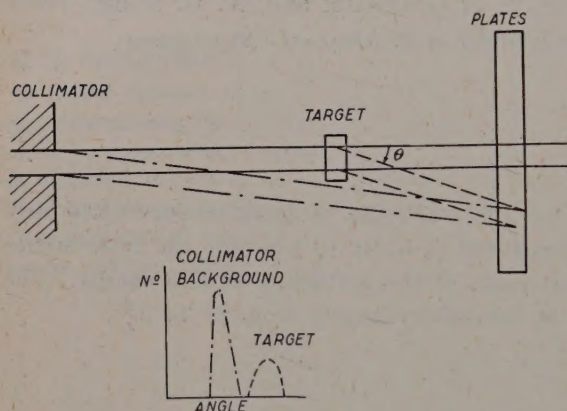


Fig. 2. - Target and plate geometry.

Measurements of angles to the nearest $\frac{1}{4}^\circ$ were made with respect to the unscattered beam in the centre of the plates. A clear separation of the scattered tracks from those in the larger peak proved possible at angles of scatter down to 1° . Reference to Fig. 2, shows that particles scattered from the target, at a given angle, should appear only in a certain region on each plate, of width equal to the beam width. At the smallest angles this region merged with the main beam in the plates,

Fig. 2 shows the geometrical arrangement and the way in which these two groups of tracks can be separated by measurements of the angles. Fig. 3 shows histograms of the number of tracks observed as a function of angle for typical parts of a plate, with and without the second carbon target in position. The subsidiary group of tracks due to the particles scattered by the target is clearly seen.

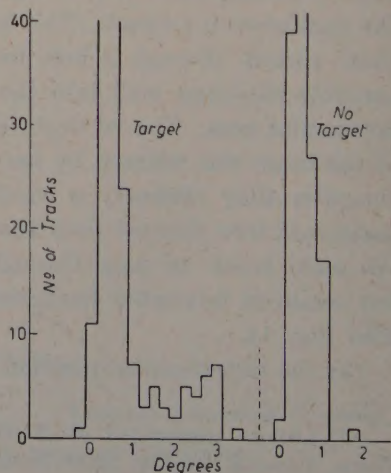


Fig. 3. - Histograms of the number of tracks as a function of angle for a certain plate area, with and without the second carbon target in position.

and observations were not possible over the whole width. A geometrical correction was therefore necessary, which was rather large for the points at 1° and $1\frac{1}{4}^\circ$ but negligible at larger angles.

4. - Results and discussion.

The values obtained for the differential cross-section are shown in Fig. 4. The errors shown are based on the standard deviations of the numbers of tracks counted. In addition to the geometrical correction, a correction was also applied to the small angle points to allow for multiple and plural Coulomb scattering. This correction was appreciable only at angles less than $1\frac{1}{2}^\circ$.

The beam monitoring was inadequate to give absolute cross-sections to better than $\pm 50\%$. Therefore, the results have been normalized to the theoretical curve shown in Fig. 4, which was obtained in the following way. The nuclear part of the scattering was calculated on the assumption that the target nucleus appears as a black disc of radius of $2.86 \cdot 10^{-13}$ cm, and it was normalized to a total elastic cross-section of 180 mb as determined by counter experiments at this energy. The Coulomb scattering was calculated exactly and the two were combined using the formula

$$\frac{d\sigma(\theta)}{d\Omega} = |R_c(\theta)|^2 + |I_c(\theta) + I_n(\theta)|^2,$$

where R_c and I_c are the real and imaginary parts of the Coulomb scattering amplitude and I_n is the purely imaginary nuclear scattering amplitude.

Within the limits of experimental error it is clear that a good fit is obtained without invoking any real part to the nuclear scattering. However, owing to the existence of the term $I_c(\theta)$ some interference does occur and makes the present results rather insensitive to a small real nuclear potential. The experiment would be more sensitive if the absolute values of the differential scattering cross-section were known and experiments to obtain this information are now in progress.

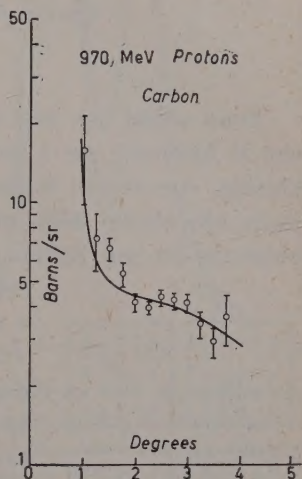


Fig. 4. - Differential scattering cross-section of 970 MeV protons from carbon at angles between 1° and $3\frac{3}{4}^\circ$; the full line is a theoretical curve (see text).

REFERENCE

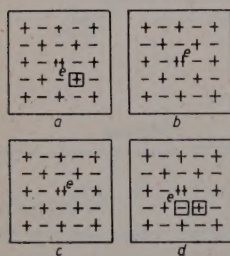
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Trasformazioni termiche dei centri F e Z_2 nei cristalli di alogenuri alcalini contenenti impurezze bivalenti positive (*).

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Sono ormai ben noti quei centri di colore caratteristici — centri Z_1 e Z_2 — che si formano con i normali metodi di colorazione nei cristalli di alogenuri alcalini contenenti impurezze bivalenti positive. Vari autori hanno contribuito allo studio delle principali proprietà di questi centri [1-5], e le ricerche condotte fin qui hanno permesso di vagliare alcuni modelli sulla natura dei centri stessi. In Fig. 1 sono illustrati i modelli di Pick e di Seitz.



- a) Z_1 centro (Pick)
- b) Z_2 centro (Pick)
- c) Z_1 centro (Seitz)
- d) Z_2 centro (Seitz)

Fig. 1.

In recenti lavori svolti dal nostro gruppo in questo campo, l'interesse era già stato portato alle trasformazioni fra centri F e centri Z , provocate mediante eccitazione ottica o termica. Si rimanda a quanto è stato pubblicato sull'argomento per maggiori dettagli [4, 5]. Oggetto della presente relazione sono alcune misure sulle conversioni termiche fra centri F e centri Z_2 , a varie temperature. Per studiare la reazione $F \rightleftharpoons Z_2$, si determinano le variazioni (provocate da annealing) nell'altezza delle bande di assorbimento F e Z_2 , come mezzo per risalire alle variazioni di concentrazione dei centri di colore interessati.

Le misure di assorbimento sono state eseguite per mezzo di uno spettrofotometro convenzionale già descritto [4], composto di: sorgente luminosa, monocromatore, cella di assorbimento, rivelatore a cellula fotoelettrica. La cella porta-cristallo fa parte di un criostato che permette di fare le osservazioni alla temperatura dell'aria liquida. Inoltre la disposizione sperimentale permette di effettuare sul posto, negli intervalli fra le misure, il riscaldamento controllato del cristallo colorato.

Si sono adoperati per le misure cristalli di KCl, cresciuti dal fuso con pic-

(*) Lavoro presentato al XLIII Congresso Nazionale della Società Italiana di Fisica, Padova-Venezia, 22-28 Settembre 1957.

cole addizioni di SrCl_2 : il tenore finale di Sr^{++} era dell'ordine di 10^{-4} molare. La colorazione per eccesso stechiometrico veniva effettuata mediante cottura in i vapori di Na a $\sim 570^\circ\text{C}$. Per uniformare campioni, e anche per rendere possibile la creazione di grosse bande Z_2 , i cristalli venivano raffreddati lentamente fino a $\sim 220^\circ\text{C}$, ivi tenuti per (15÷20) minuti, e infine portati a temperatura ambiente con un quenching veloce.

Trasformazione termica $F \rightarrow Z_2$.

Scaldando cristalli colorati con banda F (accompagnata o meno da banda Z_2 primaria) a temperature fra 80°C e 110°C , si è notata una distruzione della F con formazione e/o crescita della Z_2 . Questa conversione avviene direttamente al buio.

Le osservazioni sono state effettuate riscaldando i cristalli a temperature controllate per tempi diversi. La conversione è stata così osservata a 110°C , 100°C , 90°C . La sua velocità è ancora apprezzabile a 80°C . Un esempio delle variazioni provocate da annealing nello spettro di assorbimento è dato in Fig. 2). Da misure sulla variazione temporale del numero di centri (a 100°C) risulta che le concentrazioni di F e Z_2 tendono, per lunghi tempi di cottura, a

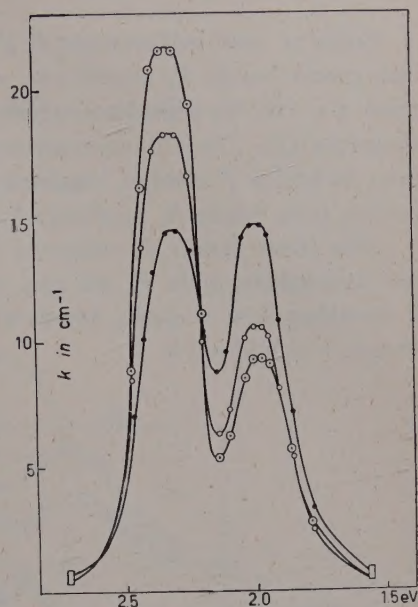


Fig. 2. — Effetto del riscaldamento a 100°C sullo spettro di assorbimento (bande F e Z_2). Osservazioni eseguite a temperatura dell'aria liquida. \odot iniziale; \circ dopo 44 minuti a 100°C ; \bullet dopo 225 minuti a 100°C .

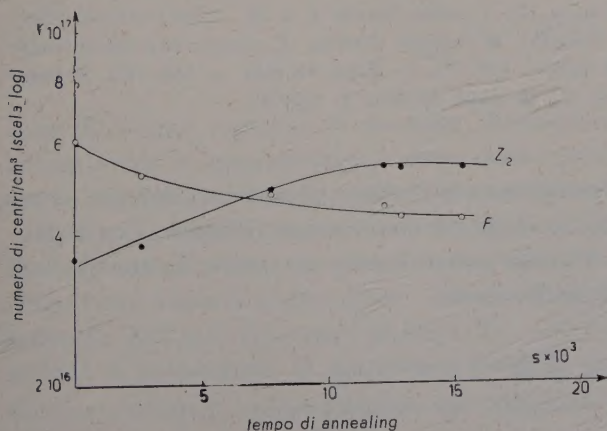


Fig. 3. — Variazione temporale del numero di centri F e Z_2 (per cm^3) in seguito ad « annealing » a 100°C .

Trasformazione termica $Z_2 \rightarrow F$.

Come si vede nell'esempio di Fig. 4, esiste anche la trasformazione inversa: una grossa banda Z_2 secondaria, creata dalla banda F per conversione termica a $\sim 105^\circ\text{C}$, regredisce quando si scalda il cristallo ad una temperatura superiore ($T = 158^\circ\text{C}$); contemporaneamente si riforma la F . Anche in questo caso, la banda F sembra raggiungere un valore di saturazione; non è chiaro, invece, se la banda Z_2 raggiunga l'equilibrio, o continui lentamente a diminuire.

Allo stesso modo si comporta la banda Z_2 secondaria di Fig. 5, ottenuta per distruzione della Z_1 ad una certa temperatura ($T = 96^\circ\text{C}$): continuando il riscaldamento a quella temperatura, la Z_2 creata « artificialmente » diminuisce, e si riforma la F .

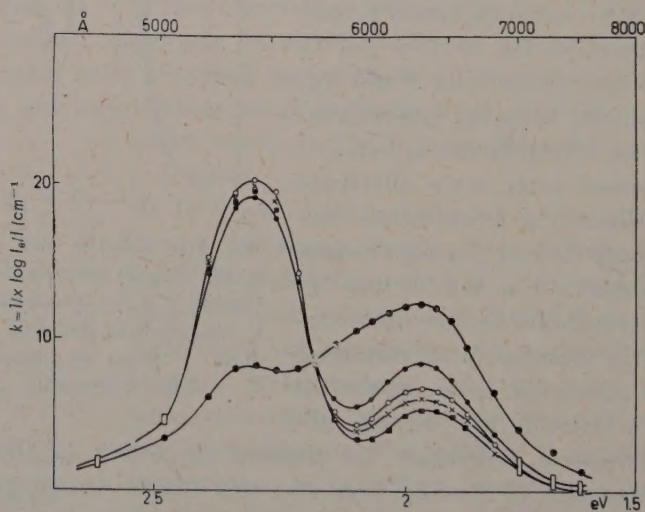
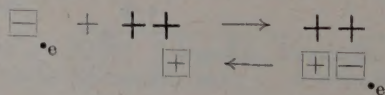


Fig. 4. - Effetto del riscaldamento a 158°C sulle bande F e Z_2 . (Osservazioni effettuate alla temperatura dell'aria liquida). ● iniziale (banda Z_2 creata per conversione dalla F a 105°C); • dopo 20 min a 158°C ; ○ dopo 40 min a 158°C ; x dopo 60 min a 158°C ; ■ dopo 80 min a 158°C .

I fatti esposti si possono interpretare in termini di una ben definita « reazione » fra centri F e centri Z_2 , analoga ad una reazione chimica. Con il modello di Seitz (v. Fig. 1) che è quello generalmente accettato, la conversione $F \rightarrow Z_2$ si può spiegare nel seguente modo:



Vale a dire, i complessi neutri (Sr^{++} -vacanza di ione positivo) migrano verso i centri F per effetto della agitazione termica, e coagulano con essi a formare Z_2 . D'altra parte, il processo è dinamico, ed esiste anche la reazione inversa per la quale i centri Z_2 si dissociano.

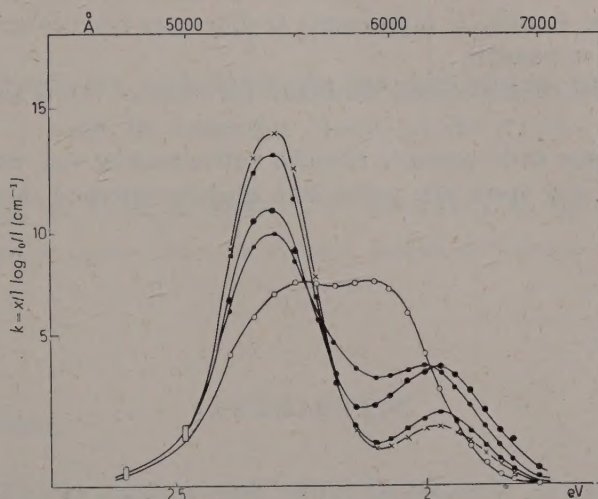


Fig. 5. — \circ iniziale (banda F e banda Z_1); \bullet dopo 16 min a $96^\circ C$ (banda F e Z_2); \bullet dopo 32 min a $96^\circ C$ (banda F e Z_2); \blacksquare dopo 107 min a $96^\circ C$ (banda F e Z_2); \times dopo 190 min a $96^\circ C$ (banda F e Z_2). (Osservazioni effettuate a temperatura dell'aria liquida.

Se quello presentato è il meccanismo giusto, si può applicare alla reazione globale la legge dell'azione di massa (usando, per i centri di colore nel cristallo, l'analogia con le soluzioni diluite):

$$(1) \quad \frac{C_F C_{++\oplus}}{C_{Z_2}} = K(T)$$

dove K è una costante di reazione dipendente dalla sola temperatura. In questo modo si spiegherebbe il verso della reazione nei diversi casi, in dipendenza dai valori relativi iniziali delle concentrazioni.

Dalla (1) si può ricavare direttamente il calore di attivazione della reazione $F \rightleftharpoons Z_2$: introducendo in essa i valori sperimentali del rapporto C_F/C_{Z_2} all'equilibrio, relativi a due diverse temperature T_1 e T_2 , si trova il valore del rapporto $K(T_1)/K(T_2) = \exp [(Q/R)((1/T_1) - (1/T_2))]$, e quindi si ricava il calore globale di attivazione, Q . La cosa è possibile poichè le $C_{++\oplus}$ incognite si eliminano: infatti la concentrazione dei complessi neutri (Sr^{++} -vacanza) è piut-

tosto insensibile a variazioni anche notevoli di temperatura, come si deduce da una conoscenza teorica dell'energia di dissociazione [6] che è circa 0.4 eV. La $C_{++}^{[+]}$ si può allora trattare come una costante su piccoli intervalli di temperatura. Il calore di attivazione così stimato, utilizzandoi valori all'equilibrio delle C per 100 °C e 158 °C, è $Q = 0.32$ eV/centro, che sembra un valore ragionevole. Come è noto, Q rappresenta la differenza fra i calori di attivazione delle reazioni componenti.

Le precedenti considerazioni sembrano far ritenere che il modello di Seitz è plausibile.

Una estensione delle presenti ricerche sull'anncaling dei centri Z_1 e Z_2 è in programma, e si spera che possa dare qualche ulteriore chiarimento.

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Distribuciones angulares de los neutrones producidos en la reacción ${}^7\text{Li}(\text{d}, \text{n}){}^8\text{Be}$ (*) (+).

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1. - Introduccion.

El estudio de la reacción ${}^7\text{Li}(\text{d}, \text{n}){}^8\text{Be}$, mediante la técnica fotográfica, ha sido objeto de una serie de trabajos anteriores [1-3] en los cuales determinamos por primera vez, algunos de los niveles energéticos del ${}^8\text{Be}$. Nuestros resultados, que coincidieron con los obtenidos independientemente por otros Centros [4, 5] fueron discutidos y puestos en duda en la Conferencia de Glasgow en 1954 [6]; de ello surgió el proyecto de una nueva experiencia para tratar de confirmar la existencia de estos nuevos posibles niveles del ${}^8\text{Be}$ [7].

Esta experiencia fué planeada juntamente con el Dr. GIBSON de la Universidad de Belfast, y el Grupo de Física Nuclear de Aldermaston, siendo realizada en dicho Centro a principios del año 1955. Los neutrones producidos al bombardear un blanco de ${}^7\text{Li}$, isotópicamente puro, con deuteronos de 1080 keV, acelerados en el Van der Graaf de Aldermaston, fueron recogidos en un lote de 12 placas fotonucleares Ilford C-2, lo que nos permitió realizar un estudio de la distribución angular de los neutrones producidos en dicha reacción correspondientes a los distintos estados excitados del ${}^8\text{Be}$.

Las placas impresionadas, una vez reveladas, fueron examinadas en los Laboratorios de la Sección de Valencia del Instituto de Optica «Daza de Valdés».

(*) Este trabajo ha sido subvencionado por la Junta de Energía Nuclear Española.

(+) Lavoro presentato al XLIII Congresso Nazionale della Società Italiana di Fisica, Padova-Venezia, 22-28 Settembre 1957.

2. - Proceso experimental.

La cámara de exposición y los datos experimentales con que fué realizada han sido descritos en [7]. Las doce placas estaban situadas alrededor del blanco a intervalos angulares de 15° y a una distancia de 16 cm del mismo. El examen de las placas se realizó mediante dos microscopios C.T.S. M4000, estudiando

un promedio de 1500 trazas de protones de retroceso en cada una de ellas, según el método descrito por POWELL [8] y RICHARDS [9].

Para el cálculo de la energía de los neutrones, utilizamos un sistema original de nomogramas, cuya descripción puede verse en [10], sin que fuera afectado por ello la precisión de la medidas.

Para obtener la distribución espectral correcta del número de neutrones, fué necesario multiplicar el número de trazas observadas correspondientes a

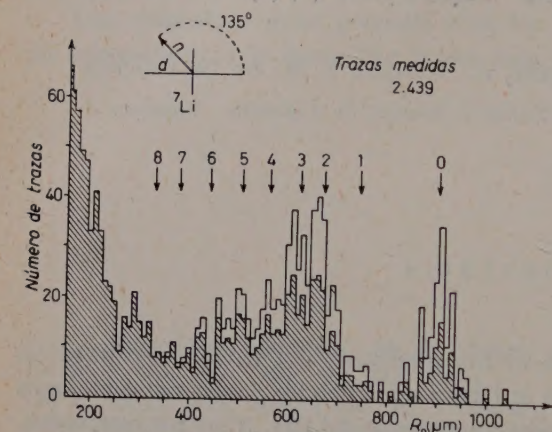


Fig. 1.

cada energía, por un factor inversamente proporcional a la sección eficaz de difusión (n-p) [11] y a la fracción de dichas trazas que no escaparon del seno de la emulsión [12]. Con ello se ha obtenido un histograma para cada ángulo como el representado en la Fig. 1 correspondiente al ángulo de 135° , en donde pueden verse grupos homogéneos correspondientes al estado fundamental y a los siguientes estados de excitación del ^6Be : 1.37, 2.09, 2.81, 3.52, 4.20, 5.01, 5.91 y 6.68 MeV.

3. - Secciones eficaces elementales.

El cálculo de las secciones eficaces elementales correspondientes al estado fundamental y a cada uno de los niveles energéticos citados, se ha realizado, siguiendo el método descrito en [13] a partir de las condiciones geométricas de la exposición, ángulo y distancia al blanco de la zona de emulsión estudiada y número total de neutrones correspondientes a cada máximo.

Los resultados, referidos en coordenadas C.M., vienen representados gráficamente en la Fig. 2.

Se han incluido en una sola distribución angular, los neutrones correspondientes al triplete 2.09, 2.81, 3.52 MeV, pues, como puede verse en la Fig. 1, los máximos correspondientes se solapan en sus bases de tal modo que es imposible el cómputo de los neutrones pertenecientes a cada uno de estos niveles.

4. - Secciones eficaces totales.

La sección eficaz total de la reacción ${}^7\text{Li}(d, n){}^8\text{Be}$, correspondiente a cada uno de los niveles de excitación citados, ha sido calculada a partir de la distribución angular y ha dado los resultados expresados en la Tabla I.

Como resultado de este trabajo, confirmamos la presencia de los niveles del ${}^8\text{Be}$ claramente establecidos con anterioridad en 2.8, 4.2 y 5.0 MeV. Asi-

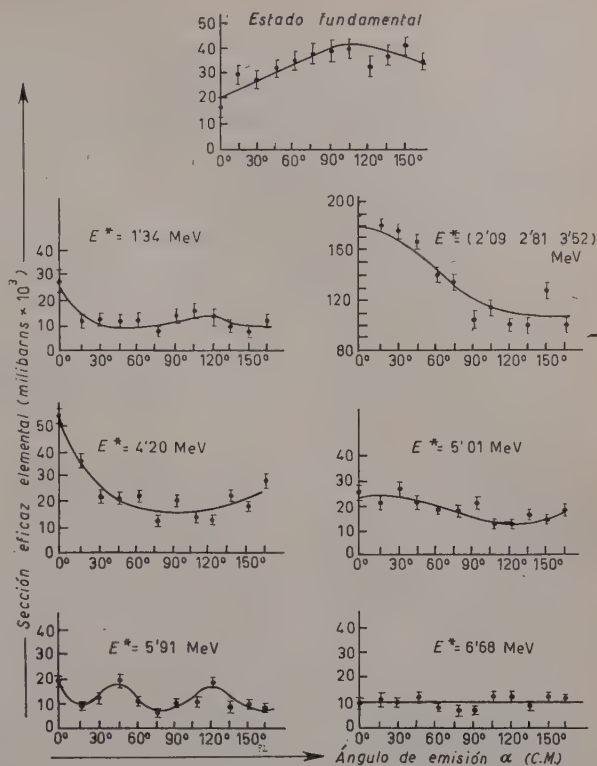


Fig. 2.

TABLA I. - Secciones eficaces totales de la reacción ${}^7\text{Li}(d, n){}^8\text{Be}$ correspondientes a los distintos niveles de excitación del ${}^8\text{Be}$.

Nivel	Sección eficaz total
Estado fundamental	0.45 ± 0.06 mb
1.37 MeV	0.16 ± 0.05 »
2.09 »	—
2.82 »	1.56 ± 0.14 »
3.52 »	
4.20 »	0.23 ± 0.05 »
5.01 »	0.23 ± 0.05 »
5.91 »	0.16 ± 0.04 »
6.68 »	0.14 ± 0.04

mismo acusamos nuevamente la presencia de los niveles 2.09 y 3.52 MeV mencionados por primera vez en un trabajo anterior [1] y que fueron citados también por otros investigadores [4, 5]. Análogamente señalamos la presencia de un nivel a 5.91 MeV citado con anterioridad por GOWARD *et al.* [14], y de 6.68 MeV mencionado por ERDÖS *et al.* [5].

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Strumento per la misura della contaminazione dell'aria dovuta a pulviscolo contenente uranio, insensibile alla radioattività dell'aria (*).

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Il monitore che descriviamo è stato studiato per il reparto di Metallurgia dell'uranio del CISE e serve a controllare durante la fase di lavorazione la contaminazione dell'aria dovuta a pulviscolo contenente uranio. Lo strumento è a funzionamento continuo ed è sensibile ad una concentrazione inferiore alla massima permessibile per esposizione professionale.

Come è noto la maggiore difficoltà nel rivelare tracce di uranio in aria è data dalla presenza di emanazione in quantità molto variabile con le condizioni meteorologiche e la ventilazione dell'ambiente. L'attività α dei prodotti di decadimento dell'emanazione è in genere maggiore rispetto a quella della concentrazione massima permessibile di uranio e pertanto maschera l'attività α dell'uranio.

La discriminazione fra le due attività viene fatta seguendo il decadimento del deposito attivo raccolto su carta da filtro mediante aspirazione, oppure aspettando che l'attività del RaC' sia completamente decaduta (ossia qualche ora). Un altro metodo usato per discriminare un emettitore α a vita lunga dai prodotti dell'emanazione, senza dover aspettare che la loro attività sia decaduta, consiste nella misura energetica delle particelle α emesse dal deposito attivo raccolto mediante precipitazione elettrostatica su un nastro trasportatore di alluminio. La difficoltà di questo procedimento consiste nel disporre di un rivelatore che dia impulsi proporzionali all'energia delle particelle α . La camera di ionizzazione a impulsi, che fornisce un'ottima risoluzione in energia, non si presta a funzionare con continuità.

Come rivelatore è stato talvolta usato il cristallo di CsI^* sotto forma di lamina sottile che dà una buona risoluzione (circa 10%) ed è pressochè insensibile all'umidità dell'aria.

(*) Lavoro presentato al XLIII Congresso Nazionale della Società Italiana di Fisica, Padova-Venezia, 22-28 Settembre 1957.

Con la discriminazione energetica delle particelle α , oltre al discriminatore ad uno o due canali, è necessaria una buona stabilizzazione della tensione di alimentazione del fotomoltiplicatore. Un'altra difficoltà non trascurabile consiste nel disporre di un rivelatore di CsI^* avente una superficie convenientemente estesa.

Il monitor è rappresentato schematicamente nella Fig. 1. Il deposito attivo viene raccolto su un nastro di alluminio che passa con continuità attraverso un precipitatore elettrostatico e fra due scintillatori di ZnS^* .

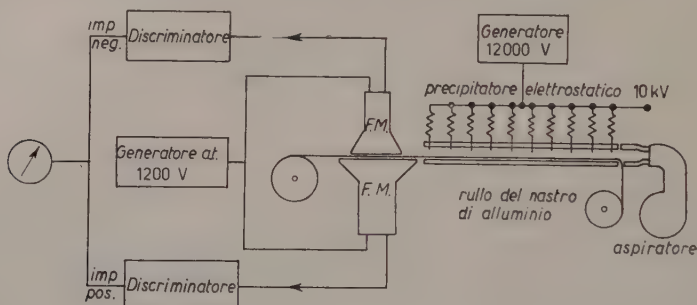


Fig. 1. — Schema di funzionamento del monitor.

Il nastro di alluminio ha uno spessore di circa 4 mg/cm^2 , tale cioè da assorbire quasi totalmente le particelle α dell'uranio e circa il 40% delle particelle α del RaC' (che rappresenta quasi tutta l'attività dovuta ai prodotti di decadimento del radon).

Il rivelatore che esplora direttamente la faccia con il deposito ha un diametro di 8 cm, l'altro ha un diametro di 12 cm. In queste condizioni i due contatori danno un numero uguale di impulsi (fluttuazioni statistiche a parte) in presenza di sola contaminazione dovuta ad emanazione. Se il deposito raccolto contiene anche attività α dovuta ad uranio questa verrà rivelata solo da uno dei due contatori. Gli impulsi dei due contatori vengono inviati ad un frequenzimetro differenziale sensibile solo alla differenza della frequenza di impulsi dei due canali di misura, che segnerà solo l'attività α dell'uranio.

Con un flusso di 500 litri di aria al minuto attraverso al precipitatore ed una velocità del nastro di 0.5 cm/min il deposito esplorato dal contatore destinato a misurare l'attività dell'uranio è quello contenuto in 5.5 m^3 di aria: l'attività α relativa ad una concentrazione massima permissibile per esposizione professionale è di circa 500 imp/min . L'attività α dovuta al RaC' rivelata dai due contatori è dell'ordine del migliaio in un ambiente moderatamente ventilato.

Con un periodo di integrazione del frequenzimetro di 40 secondi, le flut-

tuazioni statistiche dovute all'attività del RaC' sono inferiori a $1/5$ del valore relativo ad una dose permissibile di uranio.

La Fig. 2 mostra lo schema di funzionamento del frequenzimetro a transistori: la tensione è fornita da una pila di 6 volt.

Il frequenzimetro è costituito da due discriminatori ad oscillatore bloccato, direttamente alimentati dai fototubi, e da un voltmetro.

I due discriminatori forniscono impulsi di area costante uguale per entrambi, perciò le tensioni medie di uscita sono proporzionali alla frequenza degli impulsi e la costante di proporzionalità è uguale per i due discriminatori.

Il voltmetro è connesso in modo da misurare la media della differenza fra le tensioni di uscita e quindi dà direttamente la differenza fra le frequenze di impulsi nei due canali.

I discriminatori ad oscillatore bloccato sono del tipo a reazione interrotta da un diodo. Il transistor è sempre in conduzione con corrente di emettitore stabilizzata. Il diodo che interrompe la reazione è in serie al circuito di base ed è mantenuto interdetto da una tensione di blocco prefissabile.

La corrente di uscita del fototubo è inviata alla base del transistor, viene amplificata da questo e determina una tensione indotta ai capi dell'avvolgimento di base che tende a neutralizzare la tensione di blocco del diodo; quando questo diventa conduttore il circuito di reazione si chiude e viene generato un impulso che termina per saturazione del ferro del trasformatore.

Il circuito richiede due compensazioni. La prima serve a mantenere indipendente dalla frequenza degli impulsi la corrente di riposo dell'emettitore, ed è data da un transistor ausiliario che fornisce all'emettitore la corrente durante gli impulsi stessi.

La seconda ha lo scopo di eliminare le variazioni con la temperatura della soglia di scatto, imputabili alle variazioni della caratteristica del diodo di blocco della reazione e della tensione di entrata del transistor che corrisponde al valore di corrente nel circuito di reazione per cui il discriminatore scatta. La

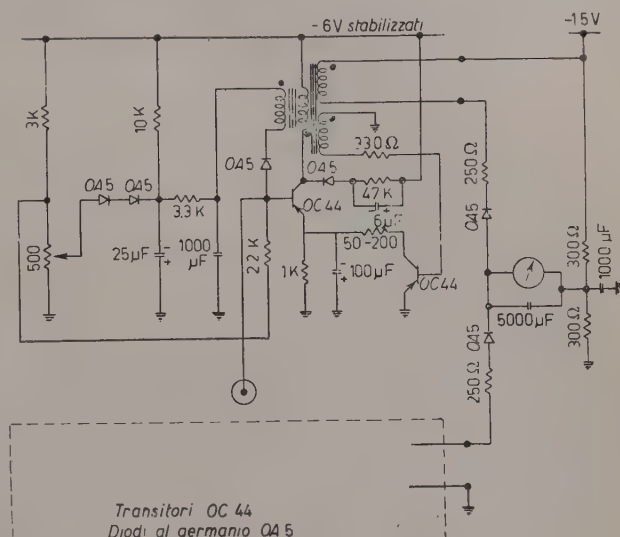


Fig. 2. — Schema elettrico del frequenzimetro.

compensazione è ottenuta aggiungendo alla tensione di soglia le tensioni ai capi di due diodi in serie tenuti in conduzione a corrente costante.

Gli impulsi di uscita degli oscillatori bloccati durano $150 \mu\text{s}$. Il discriminatore deve essere comandato da un generatore ad alta impedenza e quindi è comandato da corrente. La sua sensibilità riportata in tensione è pari ad un impulso di 50 mV ai capi di un condensatore di 20 pF .

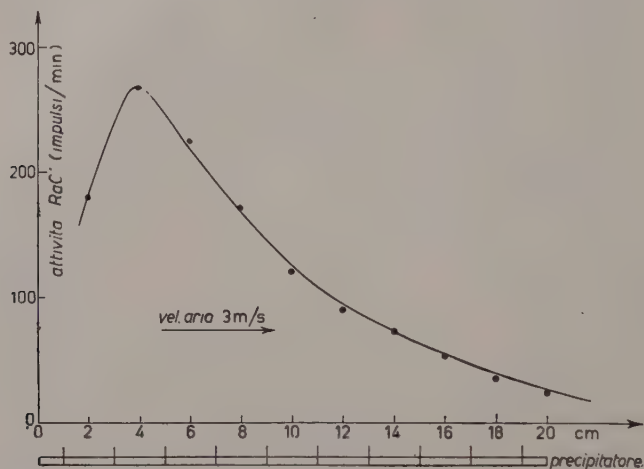


Fig. 3. - Attività del deposito raccolto in funzione della distanza dall'ingresso dell'aria.

Pure l'alta tensione dei fotomoltiplicatori è a transistori. Il precipitatore elettrostatico è formato da 50 punte distanti circa 1 cm dal nastro di alluminio tenute alla tensione di circa 10 kV da un alimentatore a transistori.

L'efficienza di raccolta del precipitatore elettrostatico si è dimostrata abbastanza buona.

Nella Fig. 3 è rappresentato il grafico dell'attività dovuta al RaC' del deposito raccolto in funzione della distanza dall'ingresso nel precipitatore per una velocità dell'aria di 3 m/s . Come si vede la maggior parte del deposito è raccolto nei primi centimetri del precipitatore.

Come aspiratore è stata usata nelle prove preliminari una pompa rotativa che verrà sostituita da una ventolina, alimentata a pile. Questo è possibile in quanto l'aria attraverso il precipitatore non incontra apprezzabile resistenza.

Examples of Direct Emission of β -rays after Capture of a μ -Meson by an Argon Nucleus (*).

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Among photographs of cosmic rays obtained with a Wilson cloud chamber filled with 75 atmospheres of argon, three examples have been found of particles, identified as μ -mesons, stopping in the chamber and subsequently emitting β -rays of energy approximately 2.8 MeV (see Fig. 1, 2, 3). In two of these cases the electron stops in the chamber. From the number of identified μ -e decays, the positive excess of μ -mesons, and the capture probability for negative μ -mesons in argon, it is calculated that approximately 35 stopped μ -mesons are contained in the photographs in question. It is most unlikely that the events can be normal μ -e decays since very few decay electrons fall in the energy band (0 ÷ 5) MeV. The Michel spectrum predicts that, on average, less than 0.03 such decays would be observed for the conditions of our experiment. It is also impossible that they could be examples of Auger electrons consequent on capture of the meson into an extra-nuclear orbit, since the energy of the $1S$ state of a μ -mesonic argon atom is only 1 MeV.

1. - Identification of the stopping particles.

Event 4418 (see Fig. 1). The whole sequence is π - μ -e. The appearance is completely typical of π - μ -e decays in our chamber (see Fig. 4 for a normal π - μ -e decay). In event 4418, the π -meson must have decayed in flight, since the μ -meson has a range of 1.3 cm as opposed to the normal range of 1.6 cm. In order to satisfy the dynamics of the event, the velocity of the π -meson at the point of decay must have been $\beta_i = 0.25$. The velocity of the μ -meson with 1.3 cm residual range is also 0.25. The appearances of the tracks are quite consistent with these values.

(*) Lavoro presentato al XLIII Congresso Nazionale della Società Italiana di Fisica, Padova-Venezia, 22-28 Settembre 1957.

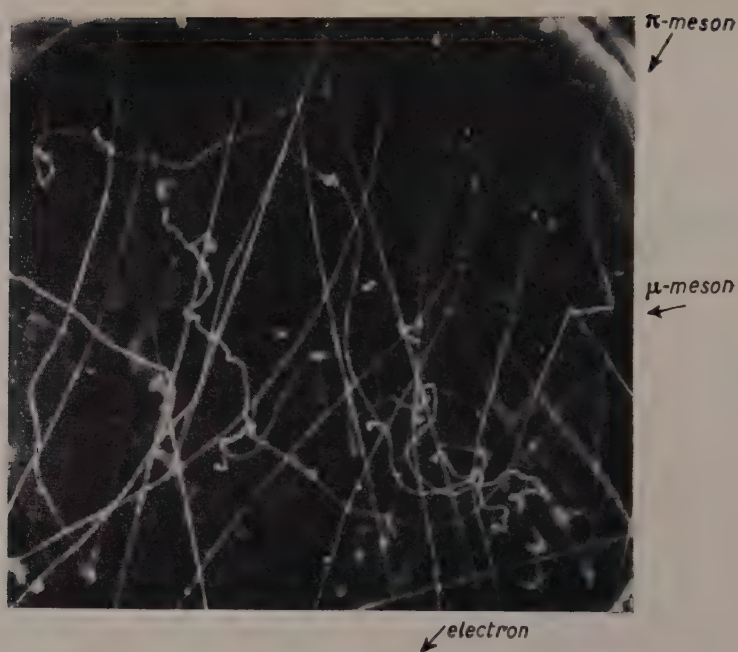


Fig. 1.



Fig. 2.

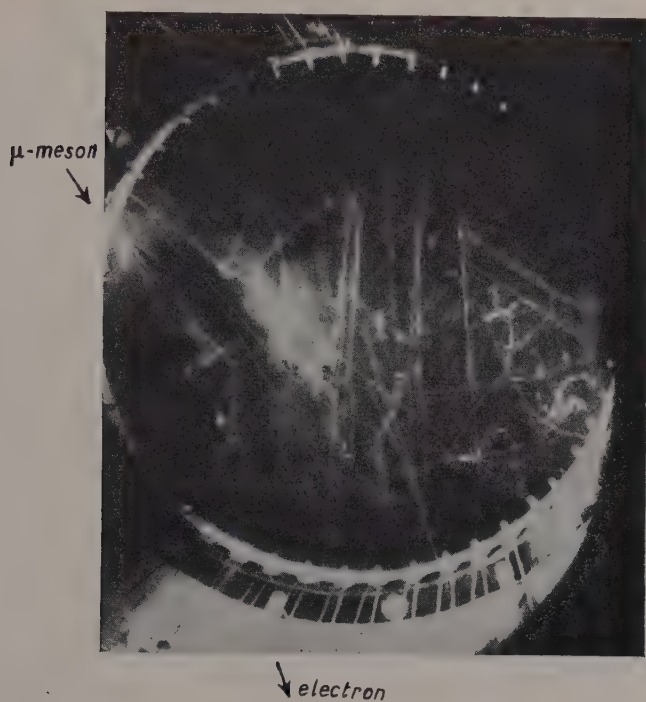


Fig. 3.

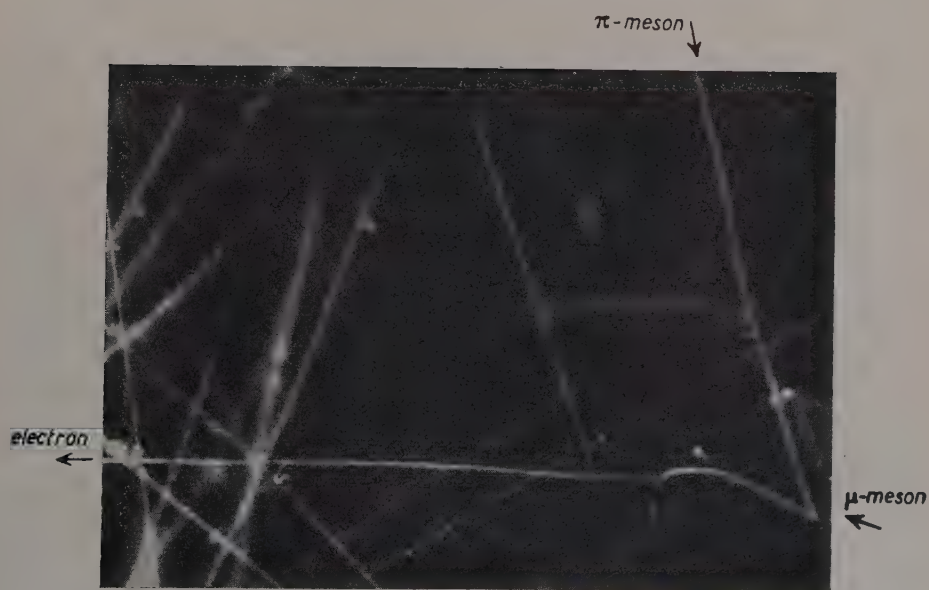


Fig. 4.

Event 4122 (see Fig. 2). It is obvious from the appearance of the event that there is a decay process involved. The minimum ionization track is an electron, from the observed scattering. The mass of the parent, determined by multiple scattering *vs.* residual range measurements, is $400^{+290}_{-150} m_e$. Since no examples of π -e decays have been found, it is identified as a μ -meson.

Event A 1414 (see Fig. 3). In this case a magnetic field of 3300 gauss was operated with the chamber and the sign of the stopping particle and the electron is known to be negative. Curvature measurements against residual range (assuming the decay process takes place at the point where there is a sharp change of curvature) give a mass value of $(150 \pm 100) m_e$. Measurements of track thicknesses are consistent with the assumption of a μ -meson coming to rest. The presence of a δ -ray 7.6 mm from the end of the track indicates that the mass cannot be greater than $250 m_e$.

TABLE I. - *Energy of the secondary particle.*

Event	Method of Measurement	Energy
4418	Range = 11.3 cm at 76 atmospheres	2.6 ± 0.1 MeV
4122	Multiple scattering	3.9 ± 0.9 MeV
A 1414	Range in chamber = 12.3 cm at 80 atmospheres	2.9 ± 0.1 MeV

2. - Interpretation of the events.

Each event is consistent with the μ -meson being negatively charged. It is therefore possible that the electron is emitted by the residual chlorine nucleus after capture of the μ -meson. In order that we could observe the β -particle, it would be necessary for it to be emitted within 200 milliseconds of the capture, and from the appearance of the electron tracks it is likely that they were emitted within 100 milliseconds. No recorded isotope of chlorine has a short enough half-life to allow the observation of the events. The shortest known half-life is that of ^{40}Cl , reported as 1.4 minutes, by MORINAGA [1] with a maximum energy of 7.5 MeV. However ^{40}Cl has a longer half-life than might be expected since its ground state has spin 2, while the ground state of argon is spin zero. The transition is therefore forbidden, and it is possible that an isomeric state of ^{40}Cl might exist with spin zero. Such a state has not been reported, but would probably not have been detected in any case. Such an isomeric state would be expected to have a much shorter half-life (between 0.15 and 15 s).

The possible consistency of the electron energies with a line spectrum raises

the possibility of them being internal conversion electrons. They cannot be interpreted in this way since the process could not compete with direct pair production to the ground state of ^{40}Cl .

3. - Conclusion.

It is concluded that the events arise from the formation of a previously unknown isomeric state of ^{40}Cl .

It is to be noticed, that if the isomeric explanation is correct it may be necessary to take this phenomenon into account in the calculations of neutron multiplicity after capture of a negative μ -meson by a nucleus.

It is intended to verify the existence of the supposed isomeric state by bombarding our cloud chamber, filled with argon, with neutrons from a Cockcroft-Walton accelerator in this department. The n-p reaction will create ^{40}Cl , and if the isomeric state exists, electron tracks, associated with the ends of the recoil protons, should be visible. It is hoped in this way to find the correct interpretation of these three events.

REFERENCE

- [1] H. MORINAGA: *Phys. Rev.*, **103**, 504 (1956).

Impiego della dipendenza della legge di decadimento di scintillatori organici dalla densità di eccitazione per discriminare neutroni veloci dal fondo γ (*).

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1. - Introduzione.

È noto che le scintillazioni prodotte in materiali organici da particelle ionizzanti hanno caratteristiche che possono dipendere in larga misura dalla natura e dall'energia della particella.

Di notevole interesse pratico sono gli effetti della densità di ionizzazione ed eccitazione sulla efficienza di scintillazione e sull'andamento dell'intensità di emissione in funzione del tempo. Tali effetti sono tra loro strettamente collegati.

Di solito ad una maggiore ionizzazione specifica corrisponde una efficienza di scintillazione minore. Ad esempio, le particelle α hanno una efficienza dell'ordine di 1/10 relativamente a β energetici; nella maggior parte degli scintillatori organici.

In generale la scintillazione incrementale dN/dE (N numero di fotoni) e la scintillazione per unità di percorso dN/dx , che sono funzioni dell'energia E della particella considerata, si possono esprimere mediante funzioni uniche di dE/dx , come è stato mostrato da TAYLOR ed altri [1].

Per interpretare la dipendenza dell'efficienza di scintillazione dalla ionizzazione specifica, occorre considerare gli effetti di quenching della fluorescenza nella scia della particella dovuti a vari processi, che è opportuno distinguere in unimolecolari e bimolecolari.

(*) Lavoro presentato al XLIII Congresso Nazionale della Società Italiana di Fisica, Padova-Venezia, 22-28 Settembre 1957.

(+) Attualmente del C.N.R.N.

Tra i processi unimolecolari, la conversione e degradazione interna di molecole eccitate è prevalente per basse densità di ionizzazione (β energetici), mentre i danneggiamenti permanenti del materiale prodotti inizialmente dalla particella hanno importanza per alte densità di ionizzazione.

I vari processi di quenching bimolecolari, dovuti ad interazioni mutue di molecole eccitate o ionizzate, sono favoriti dalla densità di queste nella scia della particella [2].

Attribuendo il quenching della fluorescenza a soli processi di tipo unimolecolare, si trova un buon accordo con la dipendenza sperimentale dell'efficienza di scintillazione da dE/dx per elettroni, protoni, deutoni, α , di varie energie, in antracene [3].

In queste ipotesi, si prevede un decadimento dell'eccitazione e della intensità di emissione di tipo esponenziale e con vita media proporzionale alla efficienza di scintillazione.

Per α e per β , ad esempio, si deduce dalle efficienze un rapporto tra le vite medie dell'ordine di $1/10$, mentre sperimentalmente le durate delle scintillazioni sono dello stesso ordine anche per particelle con alta ionizzazione specifica. Non si può render conto di questo fatto semplicemente con processi di tipo unimolecolare.

La dipendenza dal tempo della probabilità di diseccitazione delle molecole è stata da WRIGHT [4] attribuita a processi di quenching bimolecolari che, in aggiunta al quenching unimolecolare, competono, quanto più alta è la densità di eccitazione, con la emissione fluorescente.

In questo caso, per le efficienze di scintillazione si hanno risultati analoghi a quelli precedentemente citati.

L'intensità di emissione prevista ha un decadimento inizialmente rapido, che si converte, per tempi più lunghi, in un decadimento esponenziale avente la medesima costante di tempo per particelle con diversa ionizzazione specifica. Le vite medie complessive per particelle α e per elettroni risultano dello stesso ordine, in accordo qualitativo con l'esperienza.

Per osservare la rapida emissione iniziale occorrerebbe un sistema con un tempo risolutivo dell'ordine di 1 ns.

WRIGHT [5] ha studiato sperimentalmente le forme degli impulsi di scintillazione eccitati con α e con γ in cristalli di antracene, stilbene, terfenile, con un metodo indiretto basato sul fatto che la potenza generata nella resistenza di carico dell'anodo di un fotomoltiplicatore, da una serie di impulsi di scintillazione dipende dalla forma di questi.

Con una analisi dei risultati opportuna, WRIGHT ha dedotto forme di scintillazione composte di un breve picco iniziale (~ 1 ns), e di una componente a decadimento lento.

Nel caso dell'antracene questa contiene la massima parte dell'emissione, ed ha andamento esponenziale.

È assai significativo il fatto che la costante di decadimento è risultata apprezzabilmente maggiore per eccitazione con particelle α che per elettroni, rispettivamente 53 e 31 ns.

Un effetto analogo è stato rilevato anche per lo stilbene [6].

Secondo l'interpretazione di WRIGHT, il picco di fluorescenza iniziale è emesso da una piccola frazione delle molecole inizialmente eccitate che decadono rapidamente per effetto dell'alta probabilità di quenching.

La componente più lenta è emessa da una buona parte delle molecole eccitate che si formano per ricombinazione di molecole ionizzate con elettroni quando la densità di eccitazione è apprezzabilmente diminuita, e sono perciò meno soggette al quenching.

Per alta densità di eccitazione iniziale, ad esempio con particelle α , le condizioni di quenching hanno maggiore durata; poichè l'emissione è soppressa maggiormente all'inizio, risulta, oltre ad una riduzione nella efficienza, anche un allungamento nella vita media apparente della componente lenta.

2. - Parte sperimentale.

Il metodo sperimentale che descriveremo è stato progettato allo scopo di discriminare particelle che, in relazione alla ionizzazione specifica prodotta nel materiale scintillatore, diano luogo ad impulsi di scintillazione di forma differente.

Un requisito di notevole importanza pratica è che la discriminazione sia indipendente dalle ampiezze degli impulsi entro un largo campo.

Abbiamo perciò adottato il seguente principio: considerato un impulso di scintillazione, la parte A_θ di esso raccolta dall'inizio sino ad un opportuno istante θ è una frazione dell'ampiezza complessiva D , determinata unicamente

dalla forma dell'impulso. Si potrà formare con A_θ e con D una combinazione lineare che sia di ampiezza nulla per tutti gli impulsi di un dato tipo e di ampiezza qualsiasi, che possono così essere selezionati.

Nella realizzazione sperimentale (vedi Fig. 1), l'impulso di scintillazione è raccolto dal fotocatodo di un fotomoltiplicatore E.M.I. 6255 a 13 stadi di moltiplicazione.

In seguito alla moltiplicazione si raccolgono impulsi corrispondenti sull'anodo, con carica negativa, e sul penultimo dinodo, con carica positiva.

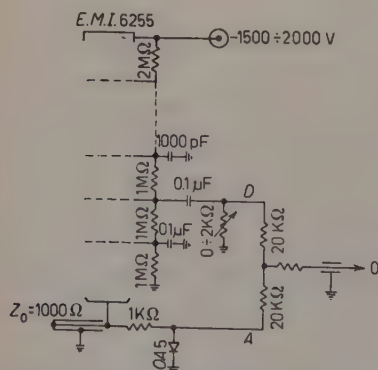


Fig. 1. - Discriminatore di forma degli impulsi.

Per evitare effetti di induzione tra questi, l'ultimo dinodo viene usato come schermo.

L'impulso di tensione sull'anodo interferisce con la propria riflessione fuori fase generata da una linea cortocircuitata, dopo un ritardo ϑ determinato dalla lunghezza della linea.

La forma dell'impulso di tensione risultante viene rettificata per mezzo di un diodo a cristallo Philips OA5, che si chiude verso massa quando la tensione raggiunge valori positivi.

Poichè la linea rimane allora chiusa su una resistenza di circa 1000 ohm, pari alla propria impedenza caratteristica, non si hanno riflessioni multiple.

L'impulso di tensione nel ramo A ha forma corrispondente all'impulso originario sull'anodo, per t da 0 a ϑ , ed è circa nullo per tempi successivi.

Dal penultimo dinodo l'impulso è prelevato mediante un condensatore di accoppiamento di $0.1 \mu\text{F}$. Attraverso la resistenza di carico variabile R_D ciascun impulso produce una tensione positiva, il cui valore integrato per tutta la durata dell'impulso è $(1/R_D) \cdot Q$, essendo Q la carica totale raccolta dal dinodo.

Gli impulsi di tensione in A e in D vengono integrati, attraverso le resistenze di $20 \text{ k}\Omega$ e le capacità parassite, per un tempo variabile dell'ordine di $0.1 \mu\text{s}$, e vengono quindi mescolati.

La resistenza variabile R_D consente di regolare il guadagno nel ramo D rispetto al ramo A .

L'impulso risultante viene formato attraverso una costante di tempo di integrazione di alcuni μs .

Gli impulsi in uscita dal discriminatore di forme sono inviati ad una comune catena di amplificazione lenta, costituita di un cathode follower e di un amplificatore modello 100, e possono essere distinti sia visualmente con l'oscilloscopio, sia mediante un discriminatore a soglia ed una scala (vedi Fig. 2).

L'apparecchiatura descritta è stata usata principalmente per distinguere impulsi di neutroni veloci di una sorgente di

Po-Be, dal fondo γ naturale o prodotto artificialmente con sorgenti di ^{137}Cs , Ra, Th.

Da queste esperienze si sono ottenute informazioni sulle caratteristiche di emissione dei vari scintillatori adoperati.

Lo scintillatore, collocato a contatto ottico con il fotocatodo, veniva irraggiato in un primo tempo con la sola sorgente di ^{137}Cs ($\sim 660 \text{ keV}$), gli impulsi risultanti venivano osservati all'oscilloscopio, e la resistenza variabile R_D era regolata finchè non si osservavano più impulsi di segno positivo.

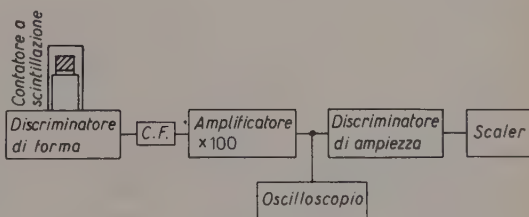


Fig. 2.

Nel circuito anodico del fotomoltiplicatore, gli impulsi che non sono almeno dell'ordine di un volt non vengono perfettamente rettificati dal diodo, tuttavia, data la particolare disposizione del circuito, se l'uscita corrispondente ad impulsi di una certa ampiezza è circa nulla, impulsi della stessa forma e di ampiezza maggiore danno uscite tutt'al più alquanto negative. Infatti anche irraggiando lo scintillatore con sorgenti di Ra ($E_{\text{max}} \simeq 1.8 \text{ MeV}$) e di Th ($E_{\text{max}} \simeq 2.6 \text{ MeV}$) non si osservano, nelle stesse condizioni, impulsi positivi.

Anche gli impulsi di fondo del fotomoltiplicatore davano uscita non positiva in quanto più brevi degli impulsi di scintillazione.

Con una piccola polarizzazione positiva del discriminatore di ampiezze si aveva, con scintillatori di piccole dimensioni ($\sim \frac{1}{2} \text{ in.}$), un conteggio di $(2 \div 3) \text{ imp/min}$, praticamente indipendente dall'irraggiamento γ , sino ad intensità corrispondenti a $(10^4 \div 10^5) \text{ imp/min}$.

In queste condizioni, irraggiando lo scintillatore con i neutroni di una sorgente di Po-Be, si osservavano impulsi positivi con ampiezza massima parec-

TABELLA I. — *Scintillatori per la discriminazione di neutroni veloci.*

Tipo	Numero di atomi H per cm^3	Vita media di scintillazione (raggi γ) ns	Forma e volume del campione	Efficienza complessiva per neutroni della sorgente di Po Be %	Efficienza di conteggio degli impulsi γ (^{137}Cs , Ra, Th) %
Antracene	$5.15 \cdot 10^{22}$	$30 \div 35$	Cubo $\frac{1}{2} \text{ in.}$ 2.05 cm^3	6.3	$\sim 1/10^4$
Trans-stilbene	$4.7 \cdot 10^{22}$	~ 6	Cubo 1 cm^3	6.5	» »
Liquido « NE 202 » (Nuclear Enterprises Ltd.)	$5 \cdot 10^{22}$	3.1	Cilindro $2 \text{ in.} \times 2 \text{ in.}$ 51.5 cm^3	13	» »
			Cilindro $\frac{1}{2} \text{ in.} \times \frac{1}{2} \text{ in.}$ 1.61 cm^3	4	» »

Altri tipi utili:

Toluene + p-Terfenile + POPOP (Nash and Thompson Ltd.).
Liquido caricato con boro « NE 311 » (Nuclear Enterprises Ltd.).
Plastico « Sintilon Brand » (National Radiac Inc.).
Polistirene + 2.5 % Terfenile + 0.03 % TPB (I.N.F.N. - Bologna).

chie volte maggiore rispetto alla polarizzazione del discriminatore, che venivano contati.

I rendimenti ottenuti nelle condizioni sperimentali più opportune, per neutroni veloci della sorgente di Po-Be con scintillatori di vario tipo e volume sono riportati nella Tab. I (*), insieme con l'elenco di altri materiali scintillatori che hanno dato risultati qualitativamente comparabili ai precedenti.

In ciascun caso le misure venivano ripetute impiegando linee con tempi di riflessione diversi, da 12.5 a 50 ns, e si ottenevano risultati dello stesso tipo.

3. - Discussione dei risultati.

Questi risultati indicano che nelle scintillazioni prodotte da protoni di rinculo la frazione di luce emessa oltre un tempo ϑ dell'ordine di qualche decina di μs è apprezzabilmente maggiore che nel caso di eccitazione con elettroni, per tutti i materiali elencati.

Nel caso dell'antracene tale fenomeno concorda in sostanza con le esperienze di Wright, precedentemente discusse.

Nel caso dello stilbene e dei vari sistemi liquidi e plastici, ai quali vengono attribuite vite medie di scintillazione più brevi di $\sim 10^{-8}$ s, gli effetti di discriminazione tra protoni ed elettroni da noi osservati implicano l'emissione di componenti lente (che durano oltre $\sim 50 \mu\text{s}$), con intensità apprezzabile almeno nelle scintillazioni eccitate da particelle densamente ionizzanti, ma non necessariamente nel caso di elettroni veloci.

L'esistenza di tali componenti lente trova riscontro nelle osservazioni di JACKSON e HARRISON [7, 8] dei decadimenti di scintillatori organici eccitati con raggi X pulsati. Date le limitazioni sperimentali, essi potevano rilevare componenti con costanti di tempo non inferiori a $0.1 \mu\text{s}$.

Per gli scintillatori cristallini (antracene, stilbene, ...) e plastici sono riportate componenti lente con vite medie poco più lunghe di $0.1 \mu\text{s}$ insieme con altre di maggiore durata (alcuni μs o più). Queste ultime sono attribuite da JACKSON e HARRISON a stati metastabili nelle molecole dello scintillatore.

Con scintillatori liquidi non è stata rilevata alcuna componente lenta (con vita media $\leq 0.1 \mu\text{s}$) di intensità valutabile, in quelle esperienze.

In un confronto con questo risultato negativo, osserviamo che nel nostro caso l'emissione lenta qualitativamente rilevata per sistemi liquidi potrebbe essere riferita, oltre che al diverso modo di eccitazione, a componenti lente con vite medie anche inferiori a $0.1 \mu\text{s}$.

(*) Questi valori sono riferiti all'intensità nominale della sorgente, che era data con $\pm 15\%$ di deviazione standard.

È verosimile che anche in scintillatori con vita media complessiva molto breve, le forme di scintillazione siano determinate in funzione della densità di ionizzazione iniziale per effetto di processi analoghi a quelli discussi nel caso dell'antracene.

Nello stilbene, ad esempio, la diffusione degli ioni, che porta alla ricombinazione di molecole in stati eccitati, non dovrebbe avere una durata sostanzialmente diversa che nell'antracene, ed è plausibile che le condizioni iniziali, create dalla particella in relazione alla sua perdita di energia specifica, intervengano con peso anche maggiore nell'equilibrio fra le quantità di luce emesse nelle successive fasi della scintillazione. Infatti una parte importante della emissione dura solamente una prima decina di ns (essendo ~ 6 ns la vita media misurata per elettroni), cosicchè quelle condizioni di quenching che agiscono prevalentemente in questa fase di alta densità di eccitazione si esauriscono in un tempo breve rispetto alle componenti lente dell'emissione, che durano oltre ~ 50 ns per protoni.

Concludiamo osservando che questi fenomeni meritano di essere studiati in modo quantitativo specialmente nei sistemi liquidi, in vista anche di nuove possibili applicazioni di questi scintillatori.

Sarebbe utile tentare di estendere il metodo da noi seguito per discriminare il fondo γ nei rivelatori a liquido caricato con boro, che sono tuttora gli unici con buona efficienza per neutroni di energia intermedia, e, in generale, per discriminare il fondo γ in misure di conteggio interno con emettitori α .

In questi casi è da prevedere una limitazione imposta dalla bassa statistica che determina le forme effettive di impulsi α di bassa energia.

Alle alte energie, la buona determinazione statistica delle forme di impulso potrebbe consentire la selezione di particelle che diano luogo a forme di emissione alquanto differenti, in relazione a diversità di ionizzazione specifica o di distribuzione della ionizzazione secondaria lungo la traccia che influiscono sui processi di quenching e di ricombinazione degli ioni.

A titolo indicativo, riportiamo che con uno scintillatore liquido cilindrico di 2 in. \times 2 in. il conteggio di fondo, in assenza di neutroni, aveva una frequenza costante superiore rispetto ad un rivelatore della stessa natura e di piccole dimensioni, e tuttavia indipendente dalle condizioni di irraggiamento anche intenso con γ di energia fino a ~ 2.6 MeV.

Tali impulsi, positivi all'uscita del discriminatore di forme e con ampiezze considerevoli, si devono presumere dovuti a particelle della radiazione cosmica, benchè non si abbiano sufficienti informazioni sperimentali sulle caratteristiche che ne determinavano la discriminazione dagli impulsi γ .

* * *

Siamo grati al prof. E. GATTI per il suo costante interessamento a questo lavoro.

Desideriamo inoltre ringraziare il sig. C. COTTINI per la sua apprezzata collaborazione.

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Rilassamento nucleare nei liquidi a varie temperature (*).

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Si riferisce su risultati di misure del tempo di rilassamento termico T_1 dei protoni in alcuni liquidi a varie temperature.

Il metodo di misura si basa sull'osservazione dei segnali dovuti a passaggi rapidi adiabatici dei nuclei attraverso la risonanza. Il dispositivo sperimentale, già descritto altrove [1], è stato da noi perfezionato allo scopo di poter variare e misurare con sufficiente precisione la temperatura del campione in esame.

Per la variazione della temperatura noi usiamo una corrente di azoto, la quale viene fatta passare attraverso un riscaldatore o un refrigeratore e viene quindi inviata nella stretta intercapedine compresa tra la parete cilindrica della fiala contenente il liquido e il supporto della bobina ricevente. La misura della temperatura viene effettuata per mezzo di una termocoppia rame-costantana, una saldatura della quale, opportunamente protetta, si trova nell'interno della fiala. La precisione nella lettura della temperatura è dell'ordine di 1/10 di grado; le differenze di temperatura fra i diversi punti del campione risultano dello stesso ordine di grandezza, se si fa in modo che le variazioni della temperatura siano abbastanza lente.

Col metodo ora descritto noi abbiamo fino a questo momento misurato il tempo di rilassamento dell'acqua per un intervallo di temperatura tra -10°C e $+90^{\circ}\text{C}$ circa e quello dell'alcool etilico per un intervallo di temperatura tra $+25^{\circ}\text{C}$ e $+70^{\circ}\text{C}$ circa. I risultati sono esposti nelle Fig. 1 e 2.

Dall'esame della fig. 1 si può osservare che il tempo di rilassamento dell'acqua da noi misurato per una temperatura di 27°C (300°K) corrisponde al valore di 3.4 s calcolato per la stessa temperatura dalla teoria di Bloembergen, Purcell e Pound [2, 3] con una precisione forse superiore a quella che ci si potrebbe attendere.

In generale, noi possiamo aspettarci che la teoria dia risultati in accordo

(*) Lavoro presentato al XLIII Congresso Nazionale della Società Italiana di Fisica, Padova-Venezia, 22-28 Settembre 1957.

con l'esperienza solo nei casi in cui il modello di Debye, adottato per la descrizione dei movimenti delle molecole, rappresenti una sufficientemente buona approssimazione.

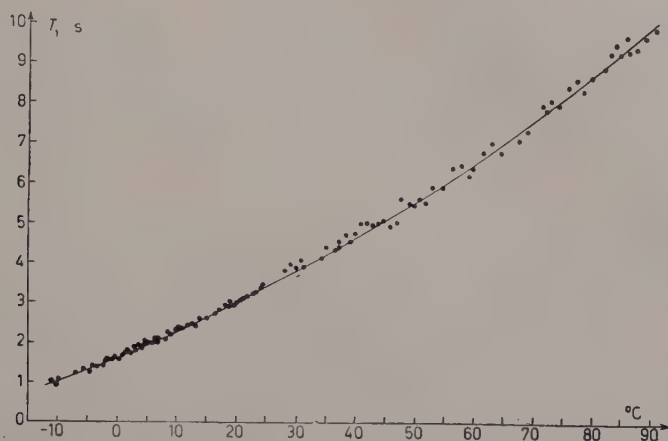


Fig. 1. - Tempo di rilassamento T_1 dell'acqua a varie temperature.

In particolare, in base alla teoria ci si può attendere che la quantità $T_1\eta/T$ (η = viscosità, T = temperatura assoluta) sia indipendente dalla temperatura. Le Fig. 3 e 4 mostrano che, per l'acqua e per l'alcool etilico, si hanno sensibili

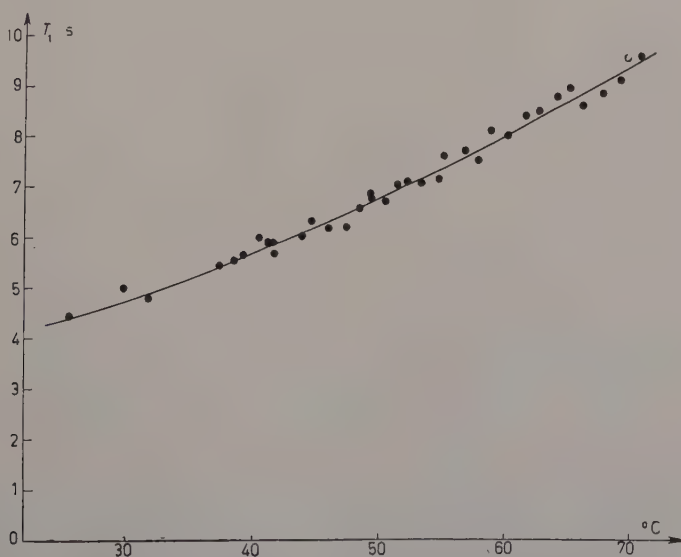


Fig. 2. - Tempo di rilassamento T_1 dell'alcool etilico a varie temperature.

deviazioni da questa legge. Il fatto che $T_1\eta/T$ diminuisce al crescere di T non sembra in accordo con l'ipotesi che in questi liquidi esistano ben definite asso-

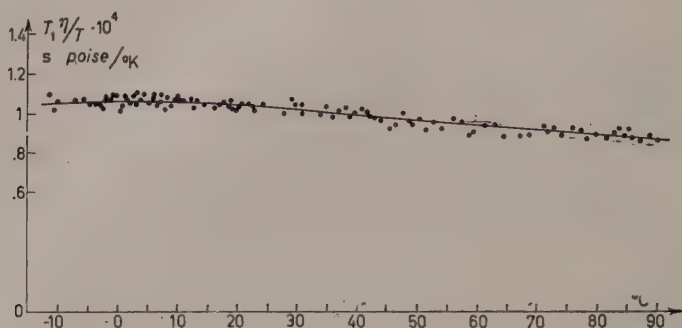


Fig. 3. — $T_1\eta/T$ per l'acqua a varie temperature.

ciazioni molecolari, ma piuttosto con l'ipotesi dell'esistenza di una struttura quasi-cristallina in progressiva distruzione al crescere della temperatura.

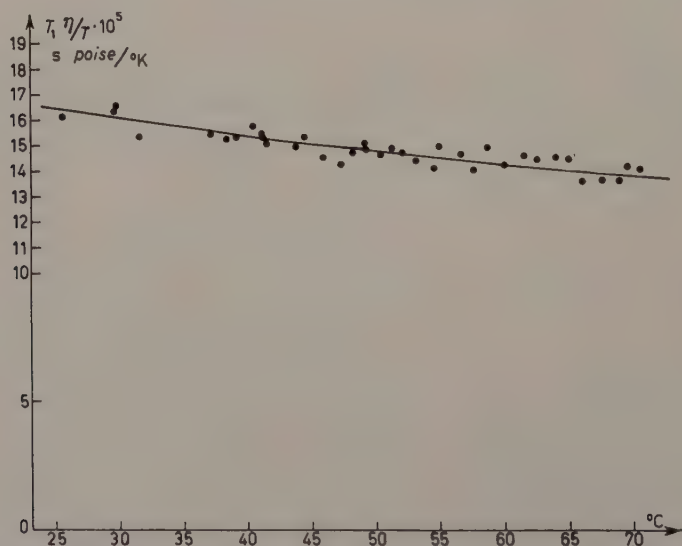


Fig. 4. — $T_1\eta/T$ per l'alcool etilico a varie temperature.

Nel caso dell'acqua si può notare che $T_1\eta/T$ si mantiene circa costante nella regione tra -10°C e $+10^\circ\text{C}$. Il confronto con altri liquidi in prossimità del punto di fusione potrà permettere di stabilire se questo andamento è da attribuirsi al particolare comportamento dell'acqua in prossimità di $+4^\circ\text{C}$.

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Stato attuale dei lavori per l'elettrosincrotrone Nazionale da 1000 MeV (*).

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1. - Stato attuale dei lavori.

Già in occasione dell'ultimo congresso della Società Italiana di Fisica a Torino abbiamo dato particolari sul nostro progetto [1], ed in questa relazione passiamo quindi in rassegna l'attuale situazione dei lavori, sottolineando alcuni contributi particolari su determinati problemi. Riferiamo poi brevemente sulla attuale situazione per la preparazione delle esperienze.

Nella tabella I è riportata la attuale distribuzione dei compiti tra i fisici e

TABELLA I. - Attuale distribuzione dei compiti tra i fisici

	Magnete e misure magnetiche	Radiofrequenza ed elettronica	Vuoto e ciambella	Iniettore	Gruppo teorico
	AMMAN BOLOGNA DIAMBRINI GHIGO SACERDOTI SALVINI SANNA	ALBERIGI MASSAROTTI MONTELATICI PUGLISI QUERCIA	CORAZZA SIRCANA	AGENO BIZZARRI QUERZOLI REALE	BERNARDINI SONA TURRIN
(L)	7	5	2	4	3
(T)	7	11	2	—	—

(*) Lavoro presentato al XLIII Congresso Nazionale della Società Italiana di Fisica, Padova-Venezia, 22-28 Settembre 1957.

gli ingegneri dei Laboratori di Frascati. Il personale della Sezione Acceleratore al completo si è trasferito dall'Istituto di Fisica di Roma ai Laboratori di Frascati nel mese di Giugno di quest'anno. Attualmente si può considerare superata la fase di trasloco e di primo ambientamento, anche se buona parte dei laboratori e dei servizi sono alloggiati in sedi non definitive, in attesa dell'ultimazione della-
laboratorio a tre piani
attualmente in costru-
zione.

Magnete.

Per i dati caratte-
ristici sul magnete ri-
mandiamo alle nostre
pubblicazioni [1].

In Fig. 1 si vede
la sala dell'edificio per
il Sincrotrone (SCAC-
CIA), nella quale è posto il magnete e dove si svolgeranno le esperienze.

In Fig. 2 è riportata una sezione del nostro magnete (SACERDOTI) limitata-
mente ai particolari che questa relazione permette.

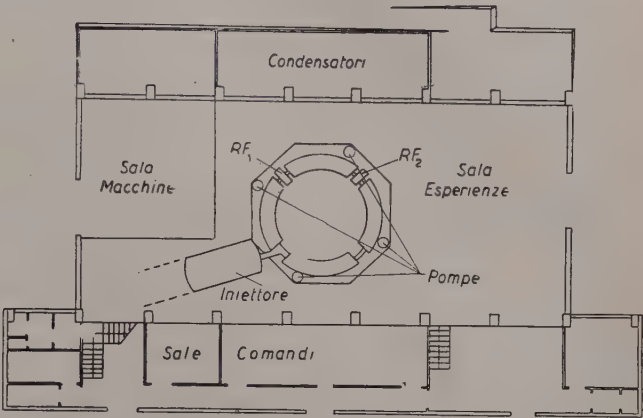


Fig. 1.

gli ingegneri dei Laboratori di Frascati.

Misura del fascio	Lavori edili e attrezzatura laboratori	Magneti per esperienze	Liquefazione idrogeno	Officina	Segreteria e servizi
CIALDEA MURTAS	CERCHIA LADU SCACCIA	BENEVENTANO TOSCHI	CARERI MONETI	—	AGOSTINI
2 —	3 2	2 —	2 1	— 11	1 11
Comlessivamente: Laureati (L): 31 Tecnici (T): 45 76					

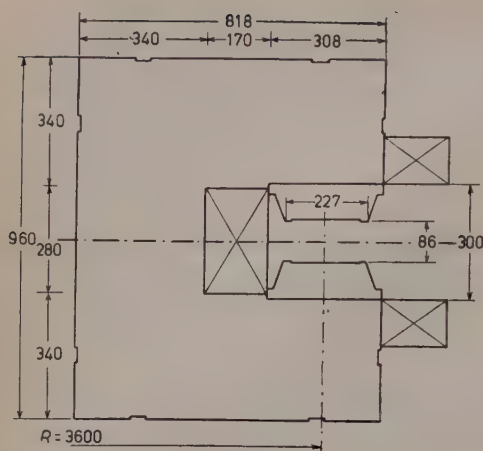


Fig. 2.

È stata ultimata la costruzione dei 64 blocchi di lamierini (ciascuno di 30 cm di dimensione azimutale) che costituiscono il nucleo del magnete. I blocchi sono stati collaudati geometricamente e misurati magneticamente presso la ditta costruttrice ed è iniziato il montaggio del magnete (GHIGO). È ultimata anche la costruzione delle espansioni polari.

È arrivata ai Laboratori la prima delle quattro bobine necessarie per i quattro quadranti. Essa è stata preparata con la normale tecnica di compoundatura, oltre che ulte-

riormente nastrata a freddo nell'insieme dei conduttori interni. È raffreddata ad acqua ed ha una struttura per necessità non usuale (GHIGO).

Come è noto il magnete sarà eccitato in corrente alternata (20 p/s) più corrente continua. È pertanto necessario un induttore di protezione (TOSCHI). Questo induttore è senza nucleo di ferro, in una struttura suggerita dal prof. SOMEDA dell'Università di Padova. L'induttore è attualmente in fase di impregnazione e la sua consegna è prevista entro un mese da oggi (in Fig. 3 è indicato lo schema del nostro induttore di protezione).

I banchi di condensatori per il magnete sono completamente montati.

Nei prossimi giorni si installerà il quadro

di controllo. Il gruppo convertitore è quasi completo poichè sono montati il trasformatore, i due gruppi rotanti, gli interruttori, le sbarre di collegamento ed il quadro (AMMAN). Dovranno ancora iniziare la messa a punto dei regolatori e le prove delle macchine su un carico resistivo, che dureranno presumibilmente circa tre settimane (AMMAN).

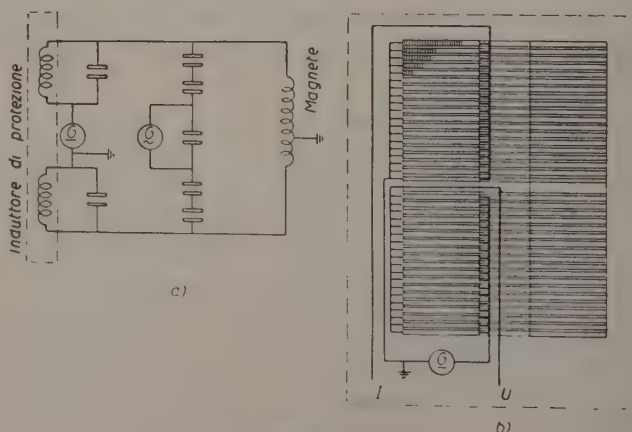


Fig. 3.

A magnete completamente montato vi sarà una ulteriore messa a punto dei regolatori sul carico reale.

Bobine di correzione.

Si è iniziata la costruzione definitiva delle varie bobine di correzione che serviranno per la definizione fina del campo magnetico (AMMAN, CORAZZA, SANNA). Queste bobine hanno chiesto uno studio piuttosto elaborato, e per questo è stato montato a Frascati il pezzo di magnete della lunghezza di un metro costruito a suo tempo. Su questo magnete sono stati sperimentati in eccitazione continua gli effetti dei circuiti di alimentazione della regione focalizzante del campo e si è giunti, dopo numerose prove, alla selezione di una serie conveniente di fili ed a valori di correzione ormai quasi definitivi [2].

Sulla base dei risultati sperimentali ottenuti in corrente continua è stata precisata l'alimentazione di questi circuiti ed è stata quindi avviata la costruzione dell'alimentazione occorrente per tutto il magnete.

Attraverso numerose serie di misure sono state poi effettuate opportune scelte di insieme dei fili che realizzano le correzioni di n di tipo prefissato anche in corrente alternata. È stata collaudata con successo l'alimentazione a questo scopo progettata e costruita.

In Fig. 4 diamo i risultati delle misure in c.a. del gradiente n del campo su tutta la profondità della gap quando non vi sono bobine di correzione (curva 1), e per diverse correnti (con un sistema di 8 fili appoggiati alle espansioni polari), che hanno lo scopo di ruotare, nel modo voluto e indicato nel diagramma, il valore del gradiente.

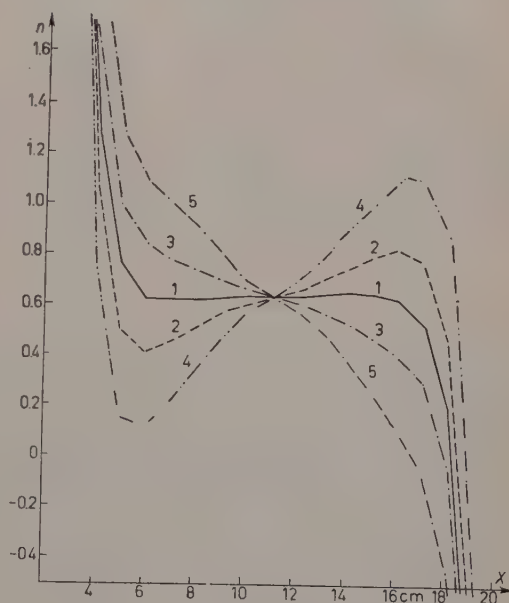


Fig. 4.

Misure magnetiche.

È stato ultimato, come si è detto, il nostro programma di misure magnetiche su ciascuno dei 64 blocchi che costituiscono il magnete (BOLOGNA, DIAM-

BRINI, GHIGO). Queste misure sono state eseguite al campo di iniezione, a campo residuo, a campo medio ed a campo massimo.

Nella Fig. 5 è riportata la distribuzione dei valori del campo massimo tra i vari blocchi, per una determinata eccitazione.

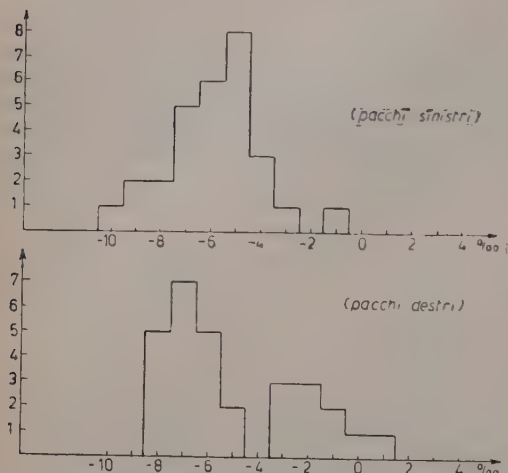


Fig. 5.

Questi risultati, considerate le nostre condizioni di eccitazione, si possono considerare soddisfacenti. I risultati così ottenuti sono stati elaborati dal nostro gruppo teorico in collaborazione col gruppo di misure magnetiche a fine di stabilire l'ordine migliore con cui i blocchi dovranno essere allineati sul Sincrotrone (BERNARDINI, SONA, TURRIN).

Si è quindi calcolata la forma dell'orbita chiusa relativa alla distribuzione dei blocchi decisa, e nell'ipotesi che le fluttuazioni azimutali di campo magnetico siano

essenzialmente dovute ai blocchi più che alle espansioni polari. La forma dell'orbita chiusa è indicata nella Fig. 6. Come si vede l'orbita chiusa si scosta dall'orbita ideale di equilibrio per non più di 2 mm a campo massimo.

Dai primi di settembre è iniziata la programmazione delle misure magne-

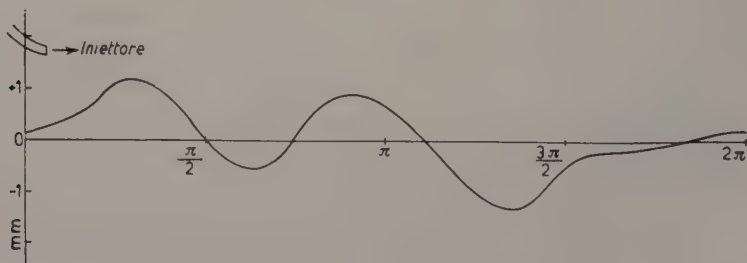


Fig. 6.

tiche sul primo quadrante del sincrotrone e si sta montando e mettendo a punto tutta l'apparecchiatura necessaria.

La nostra strumentazione per le misure magnetiche si è dimostrata soddisfacente, ed anzi può dirsi che essa è probabilmente la più avanzata oggi esistente per misure su macchine nucleari. Tra gli strumenti originali o le innovazioni apportate a queste misure, vogliamo ricordarne alcuni.

Lo strumento approntato da DIAMBRINI per le misure del piano magnetico mediano in corrente continua è ormai in uso presso di noi da più di un anno e soddisfa egregiamente alle sue funzioni. Tale strumento è già stato descritto [3].

Si è studiato un nuovo tipo di bobinetta rotante (GHI-Go) per la misura del gradiente n di campo da 500 a 10 000 gauss. L'apparecchio consiste di una bobina con l'asse magnetico ortogonale al piano magnetico mediano, e che ruota intorno ad un asse anch'esso ortogonale al piano magnetico mediano, come descritto nella Fig. 7. Il moto di rotazione

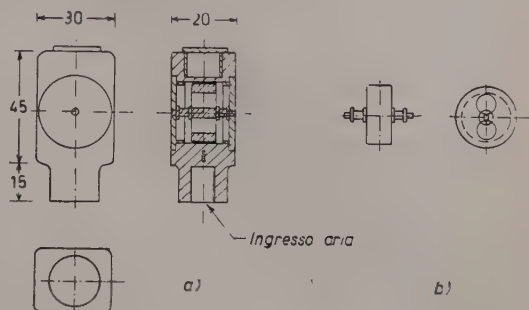


Fig. 7.

della bobina è ottenuto mediante una turbina a palette ad aria compressa. Facendo ruotare la bobina nelle condizioni di cui sopra, in un campo magnetico con gradiente b si ha una forza elettrica alternata il cui valore di picco è:

$$e = \Sigma(B_1 - B_2)2\pi v,$$

dove Σ è l'area spire della bobina e $B_1 - B_2$ sono i valori del campo ad una distanza $2r$, essendo r la distanza tra l'asse magnetico e l'asse di rotazione. La misura di $B_1 - B_2$, dalla quale si risale al valore di n , può realizzarsi indipendentemente dalla frequenza v , integrando il segnale di uscita. Lo schema a blocchi dell'apparecchio è indicato in Fig. 8.

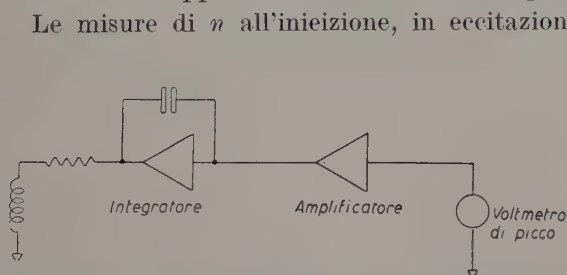


Fig. 8.

Le misure di n all'iniezione, in eccitazione alternata più continua, sono state fatte impiegando il metodo delle peaking strips [3].

Si sta studiando anche l'automazione di queste misure per guadagnare tempo intorno al magnete. Si è concluso per una ragionevole automazione che verrà a suo tempo descritta. Rite-

niamo infatti che sia conveniente pubblicare in una nota estesa la descrizione degli strumenti di misura da noi costruiti e le misure magnetiche effettuate.

Iniettore.

L'iniettore, come già noto dalle nostre relazioni [1], (pag. 1), è del tipo di Cockroft e Walton. Esso è curato dai fisici dell'Istituto Superiore di Sanità di Roma.

Il moltiplicatore di tensione è completo in tutte le sue parti. Sono in questo periodo in corso le prove di collaudo ad alta tensione del dispositivo montato nella sua tank in azoto compresso a 12 atm. Si prevede che tali prove dureranno ancora una decina di giorni dopo di che tutto il complesso sarà trasportato a Frascati per la messa in opera. L'alimentatore a 1000 periodi stabilizzato è già stato collaudato e trasportato a Frascati.

I catodi e la sorgente ed il tubo acceleratore sono pronti; la sorgente ed i singoli elementi del tubo acceleratore sono stati collaudati alla loro tensione di lavoro su di un banco da vuoto provvisorio.

Il banco da vuoto è stato automatizzato ed il suo collaudo è ora in corso. Si sta provvedendo al montaggio su di un carrello indipendente dalla pompa a diffusione.

I circuiti per il comando da terra della sorgente di elettroni sono pronti e collaudati. È in corso il montaggio definitivo dei circuiti di comando dei filamenti del catodo di comando della sorgente destinati a stare nella parte alta tensione del moltiplicatore; anche questi circuiti sono stati collaudati in montaggio di prova.

Sono state già progettate ed eseguite le lenti di focalizzazione del fascio di elettroni ed i relativi alimentatori stabilizzati in corrente. Sono progettate o in corso di esecuzione le bobine deflettrici per la centratura del fascio.

Il deflettore, insieme a tutti i meccanismi che ne permettono i movimenti è già costruito. È quasi ultimato il pannello che contiene tutti i relativi comandi. È iniziata la costruzione del giunto mobile di collegamento fra il deflettore e l'iniettore. È stato costruito in montaggio di prova un alimentatore altamente stabilizzato a 50 kV per il deflettore, che è attualmente in collaudo; contemporaneamente è stato cominciato il suo montaggio definitivo.

Vuoto e ciambella.

L'impianto di vuoto dell'elettrosincrotrone sarà pronto per il montaggio entro questo mese.

Presso i nostri laboratori è ora in via di realizzazione il pannello dei telecomandi dell'intero impianto di vuoto da montarsi sulla consolle di comando generale del Sincrotrone.

Dal 1° Settembre 1957 si è cominciata la costruzione della ciambella nelle

sue dimensioni definitive apportando al vecchio modello alcune modifiche che la rendono meccanicamente più resistente e di maggiore semplicità di montaggio (CORAZZA, SIRCANA).

Da prove fatte in un compressore idraulico la ciambella resiste senza rompersi ad una pressione di 2.5 atm. La rottura avviene a circa 3 atm od oltre.

Ogni quadrante di ciambella sarà composto di tre settori uniti fra loro

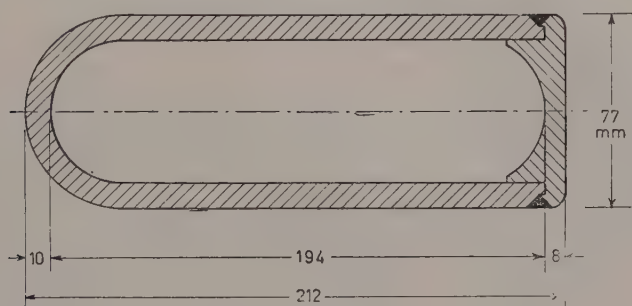


Fig. 9.

con una guarnizione a tenuta di vuoto tipo « O » ring alloggiata sulla testata di ogni settore in una gola ricavata per mezzo di un pantografo.

In ogni quadrante la ciambella ha cinque finestre trasparenti che permettono di esplorare l'interno. In prossimità di tali finestre sono sistemate bandierine fluorescenti normalmente tenute in posizione tale da non disturbare il percorso degli elettroni. Le bandierine per mezzo di un sistema meccanico a tenuta di vuoto possono essere poste in direzione normale al fascio in modo da individuarne la posizione. Tali bandierine serviranno principalmente alla ricerca del fascio nel periodo di messa a punto dell'elettrosincrotrone.

In Fig. 9 è illustrata la sezione della ciambella. La ciambella è coperta all'interno con uno strato di acciaio inossidabile dello spessore di 5/100 di mm in striscie di 12 mm distanti tra loro 3/10 di mm.

Radiofrequenza.

È stata ultimata la costruzione, messa a punto e collaudo della catena di amplificazione per la radio frequenza (RF_1) (ALBERIGI, LEPRI, PUGLISI, QUERCIA). Si sta ora precisando l'inserimento della cavità risonante nella sezione diritta della macchina ad essa riservata, ed il passaggio della ciambella attraverso di essa. Sono in costruzione i dispositivi di sicurezza e gli apparati per la messa in funzione automatica di ogni parte di questo impianto. È già stata

pubblicata in una delle nostre relazioni la descrizione particolareggiata della parte elettronica della RF_1 [4].

È stato risolto (QUERCIA [5]), con un dispositivo elettronico del quale diamo qui il principio di funzionamento, il problema di controllare la frequenza della cavità RF_1 , mentre essa varia rapidamente nel tempo: come è noto questa variazione è necessaria perchè gli elettroni all'iniezione (2.5 MeV totali) hanno una velocità iniziale inferiore del 2.1% alla velocità della luce.

La misura della frequenza deve essere fatta in un tempo sufficientemente

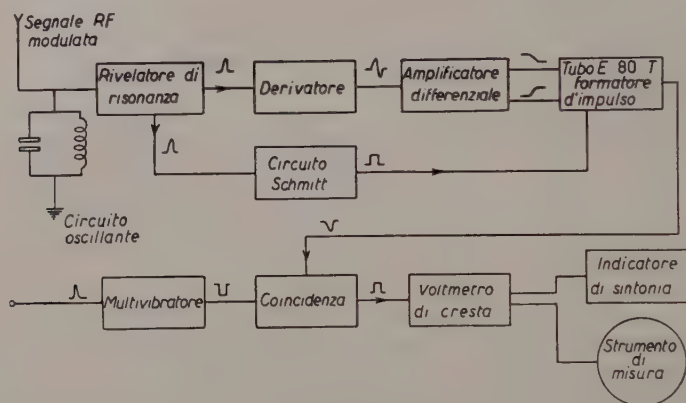


Fig. 10.

breve affinché il valore di frequenza sia sufficientemente definito (almeno entro l'uno per mille).

Questo è stato realizzato confrontando ad ogni tempo voluto, con un sistema di coincidenze, l'istante T_1 in cui la frequenza ν_1 della cavità raggiunge un certo valore e l'istante T_2 in cui il campo magnetico ha il valore che a quella frequenza deve corrispondere. I due istanti devono essere in coincidenza. La frequenza ν_1 viene direttamente prelevata mediante un segnale captato con una piccola antenna ed iniettato entro un circuito risonante avente l'opportuno fattore di merito ed accordato sulla frequenza ν_1 .

Diamo in Fig. 10 lo schema a blocchi di questo strumento.

RF_2 (cavità risonante ad alta tensione).

Gli elettroni vanno portati da circa 10 MeV a 1000 MeV per mezzo di una cavità risonante ad alta tensione (circa 60 000 V_{\max} alla gap), della quale è terminata la progettazione nelle sue linee generali ed è stata iniziata la costruzione e messa a punto delle varie parti della catena (PUGLISI, MASSAROTTI).

Lo stadio pilota (uscita 300 W alla frequenza finale) è stato costruito e collaudato; così pure il 1° stadio (uscita 1.3 kW) che però non è ancora in veste definitiva.

Lo stadio prefinale (10 kW) è parzialmente costruito e sarà provato in questi giorni; lo stadio finale (40 kW) è ancora in fase di modello in scala 1:1.

Si è costruita una cavità risonante con la regione elettrica sotto vuoto e su di essa sono stati sperimentati i metodi per evitare il multipacting elettronico. Con queste cavità preliminari si sono ottenuti circa $25\,000\text{ V}_{\text{max}}$.

È attualmente in costruzione la cavità di tipo definitivo. Diamo in Fig. 11 lo schema in sezione di questa cavità.

In un primo tempo si era considerato sufficiente una tensione massima di $(30\,000 \div 40\,000)\text{ V}$. Questa cifra è stata oggi portata a $60\,000\text{ V}$ sulla base delle esperienze fatte recentemente presso altri sincrotroni e dei risultati teorici di vari autori [6].

Il nostro gruppo teorico (C. BERNARDINI) ha calcolato la tensione di R.F. massima necessaria per la nostra macchina. La situazione è illustrata nel diagramma di Fig. 12. In questo diagramma si dà l'ampiezza massima di oscillazione di Sincrotrone, per varie tensioni massime della cavità, in funzione dell'energia degli elettroni.

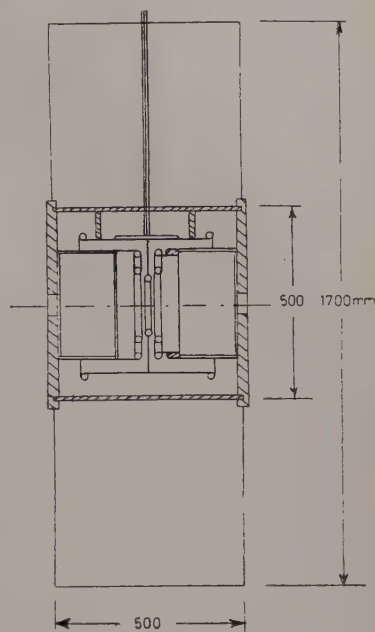


Fig. 11.

Affinchè non si perda una elevata percentuale degli elettroni è necessario che l'ampiezza massima delle oscillazioni permesse sia abbastanza più grande

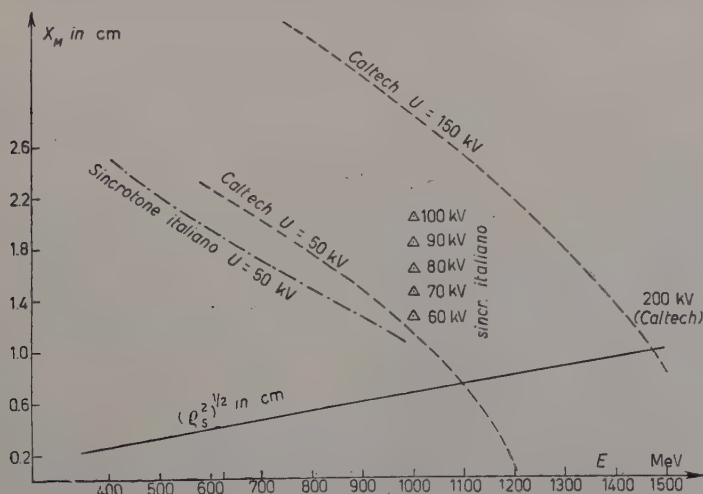


Fig. 12.

delle dimensioni quadratiche medie che il fascio raggiunge per effetto delle fluttuazioni nella perdita di irraggiamento degli elettroni.

Come si vede, questa condizione è rispettata sino a 1000 MeV solo se la cavità ha una tensione massima di almeno 50 000 V. Si stima che per il nostro elettrosincrotrone sia conveniente una cavità di almeno 60 000 V.

In questi mesi si è calcolato [7] (PUGLISI, QUERCIA) il calo di tensione massimo della cavità RF₁ per diverse correnti di elettroni circolanti entro la ciambella. Le cavità da noi costruite e la potenza dei nostri impianti permetteranno certamente di accelerare un fascio di 10^{11} elettroni senza apprezzabili perdite di tensione della cavità.

Estrazione del fascio di elettroni.

Si è fatto uno studio preliminare della possibilità di estrarre il fascio di elettroni dalla macchina allo scopo di fare esperienze con esso, oltre che con il normale fascio di fotoni. Queste ricerche hanno già indicato che questa operazione presenta varie difficoltà e non si è ancora trovata una via di sicuro successo.

In particolare (DIAMBRINI) si è esaminata l'eventualità di migliorare il rendimento di estrazione del fascio di elettroni disponendo due zone di campo magnetico alternato, una interna e l'altra esterna all'orbita stabile. In tal modo gli elettroni potranno entrare in queste regioni già con fase di risonanza e ad ogni giro. Questo meccanismo dovrebbe diminuire lo sparpagliamento degli elettroni stessi, che potranno così raggiungere il campo magnetico in maggior numero che non nel caso del metodo tradizionale di una sola regione alterata.

2. - Preparazione delle esperienze con l'elettrosincrotrone.

Lo sviluppo della preparazione delle esperienze ha proceduto secondo le indicazioni già contenute nel verbale della riunione svoltasi a Roma nel Gennaio 1957 [8].

Non riteniamo pertanto opportuno riportare in questa sede i particolari progetti in corso e ci limitiamo ad alcune osservazioni generali.

Strumenti per la misura del fascio.

Si tratta di fare una misura assoluta dell'intensità del fascio dei γ e di misurare sperimentalmente la distribuzione in energia dei fotoni che lo costi-

tuiscono. Per la misura dell'intensità si stanno costruendo (CIALDEA) gli opportuni calorimetri, che misurano l'energia totale del fascio che in essi si estingue completamente.

È inoltre in corso di progetto una camera a ionizzazione del tipo già costruito presso l'Università di Cornell [9].

Per lo spettro del fascio γ è in progetto uno spettrometro a coppia (BENEVENTANO, RISPOLI, SANNA). Il progetto del magnete per esso è particolarmente oneroso e si sta tentando una soluzione che permetta di impiegare questo magnete nelle esperienze, una volta che il suo impiego con lo spettrometro a coppie sia esaurito.

Magneti per le esperienze.

Sono attualmente in costruzione due magneti deflettori, illustrati in Fig. 13. Come si vede si è scelto un magnete con giogo a C per lasciare liberi tre lati del traferro, poichè si ritiene che questo permetta una maggior latitudine di impiego. Per questi magneti è previsto un campo massimo di 16 000 gauss, e l'area del traferro di $80 \times 40 \text{ cm}^2$ permetterà deflessioni apprezzabili sino a 1000 MeV.

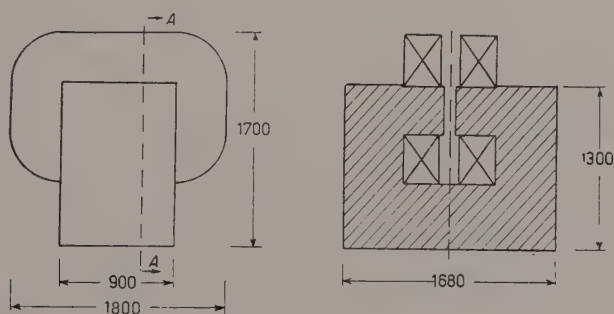


Fig. 13.

Si sono studiate le più convenienti forme polari per il foceggiamento delle particelle cariche. Sono previste espansioni polari a campo medio ed espansioni polari a foceggiamento forte.

È sempre conveniente appoggiare le attrezzature generali a determinati progetti di esperienze, anche se questi progetti, per il tempo che intercorre tra il progetto e la possibile esecuzione delle misure, dovranno essere notevolmente mutati. Pertanto, i magneti ora in costruzione sono quelli adatti ad una esperienza di fotoproduzione di mesoni carichi con un fascio γ di energia massima di 1000 MeV [10].

Alcune osservazioni sulla distribuzione dei lavori.

È importante tener presente che il Sincrotrone produrrà scientificamente non quando sarà finito ma solo quando sarà pronta insieme ad esso l'attrezzatura generale per le esperienze, oltre all'elettronica, ed ai rivelatori di varia natura relativi a determinate esperienze.

Come si è visto (cfr. tabella I) è ormai sufficiente il numero di fisici e di ingegneri che lavorano alla macchina propriamente detta, ma è certamente troppo ristretto il numero di fisici che preparano le attrezzature generali o determinate esperienze.

Dobbiamo tener presente che il nostro elettrosincrotrone è stato costruito per permettere di svolgere ricerche nucleari nel campo delle alte energie ai fisici interessati di tutte le Università italiane, e che a questa macchina, perchè la costruzione sia giustificata, dovranno avvicinarsi con continuità diversi gruppi di ricercatori per diverse esperienze.

È difficile dire quando la macchina sarà funzionante poichè inizierà tra poco quella fase di messa a punto, sulla cui durata non ci si può impegnare. Debbo dire però che è opinione anche degli esperti con i quali abbiamo scambiato che, al punto in cui oggi siamo, noi dovremmo avere un numero di fisici che si dedichino alla preparazione delle attrezzature generali e delle esperienze non molto inferiore al numero di fisici che si dedica direttamente alla macchina. Attualmente invece i fisici che dedicano la loro maggiore attività alla preparazione delle attrezzature e delle esperienze non supera il numero di sei.

È a mio parere possibile, ma non nel breve tempo ancora rimasto per questa relazione, dimostrare che i limiti di precisione della sperimentazione nucleare almeno con un elettrosincrotrone, non sono oggi tanto costituiti dalla precisione e ripetibilità della macchina quanto dalla precisione degli strumenti calibratori del fascio e dei rivelatori delle interazioni nucleari.

Sotto questo punto di vista si può dire che un elettrosincrotrone merita uno sforzo sulle attrezzature generali e sui rivelatori di ogni esperienza, maggiore di quanto non si faccia anche oggi in ogni laboratorio ove sono funzionanti elettrosincrotroni. La nostra situazione in questo senso rischia di essere anche peggior di quella di altre macchine già funzionanti.

Elettronica per le esperienze.

Da circa sei mesi è in funzione (RISPOLI) un Centro di elettronica che si occuperà particolarmente di realizzare sia gli strumenti elettronici per le esperienze, come i rivelatori propriamente detti (contatori di Čerenkov, scintillatori, ecc.).

Si è considerata questa iniziativa necessaria, dato l'alto livello di specializzazione raggiunto dall'elettronica. D'altra parte questo Centro dovrà realizzare alcuni prototipi che possano essere dati all'industria italiana per la loro moltiplicazione.

Considerato l'attuale stato dei lavori, è da augurarsi che ben presto inizi a Frascati la preparazione delle esperienze con la nostra macchina, da parte dei fisici delle nostre Università.

Uno stretto contatto tra chi fa la macchina ed i fisici che preparano esperienze è ormai, a mio giudizio, necessario.

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On The Theory of Higher Spin Fields (*).

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(ricevuto il 28 Ottobre 1957)

CONTENTS. — Introduction. — 1. The free fields. — 2. The spin projection operator. — 3. Electromagnetic interactions. — 4. The minimal condition. — 5. Wave equation for spin $\frac{3}{2}$. — 6. Algebra of the Γ -matrices. — 7. Integer spin fields. — 8. Magnetic moment of fermions. — 9. Polarization operators. — 10. Angular distributions. — 11. Appendix.

Introduction.

Recent experiments ⁽¹⁾ on strange particles have given some support for the possibility that elementary particles of spin higher than 1 may exist. In particular the observed angular correlation between the planes of production and decay of the hyperon Λ and Σ , and the anisotropy of the angular distribution of the Σ decay products, seem to indicate that the spins of these particles are $\frac{3}{2}$ or higher ⁽²⁾.

(*) Part of dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy from the University of California at Los Angeles.

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(1) W. D. WALKER and W. D. SHEPHARD: *Phys. Rev.*, **101**, 1810 (1956); W. B. FOWLER, R. P. SCHUTT, A. M. THORNDYKE and W. L. WHITEMORE: *Phys. Rev.*, **91**, 1287 (1953); **93**, 861 (1954); **98**, 121 (1955); L. W. ALVAREZ, H. BRADNER, P. FALK-VAIRANT, I. D. GOW, A. H. ROZENFELD, F. T. SOLMITZ and R. D. TRIPP: *UCRL* 3583, University of California Radiation Laboratory, Berkeley, 1956.

(2) At the time that the present work is being prepared for publication, evidence for the higher spin of the hyperons is weakening, see *Proceedings of the Rochester Conference of High Energy Physics*, 1957 (to be published).

The theory of higher spin fields proposed by FIERZ ⁽³⁾, and FIERZ and PAULI ⁽⁴⁾ has been simplified by the work of RARITA and SCHWINGER ⁽⁵⁾, GUPTA ⁽⁶⁾ and MOLDAUER and CASE ⁽⁷⁾. In spite of these simplifications, however, calculations are very lengthy, as illustrated by the calculation of the Compton scattering cross-section by MATTHEWS ⁽⁸⁾.

It is the aim of this note to present a new simple formulation of the Fierz-Pauli theory, and to analyze in some detail the structure of this theory for arbitrary spin.

All relativistic theories of free fields are based on the principle of invariance under the group of co-ordinate transformations known as the Lorentz group ⁽⁹⁾. This principle requires that the wave function form a basis for a representation of the Lorentz group. The simplest kind of field is defined by an irreducible representation and is said to describe an elementary particle, which provides an exact and natural definition of the latter concept.

The Lorentz group possesses two invariants

$$p^2 = p^\mu p_\mu, \\ p^2 S^2 = \frac{1}{2} L^{\mu\nu} L_{\mu\nu} p^2 - L^{\mu\lambda} L_{\mu\lambda} p_\nu p^\nu.$$

In an irreducible representation these must be multiples of the identity. Define, therefore, two numbers m and s , such that

$$(a) \quad p^2 = -m^2, \quad (\text{Definition of mass}), \\ (b) \quad S^2 = s(s+1), \quad (\text{Definition of spin}).$$

In all the cases of physical interest m is a positive number or zero, and $2s$ is a positive integer or zero. In these cases there is only one representation of the Lorentz group for any given set of values of m and s ⁽¹⁰⁾, apart from an ambiguity in the choice of reflection operators ⁽¹¹⁾. In the following the value $m = 0$ is excluded from consideration, unless otherwise stated.

In the first three sections conditions (a) and (b) are formulated in terms of conditions on the wave function, in the case of no external forces. A « spin

⁽³⁾ M. FIERZ and W. PAULI: *Proc. Roy. Soc.*, A **173**, 211 (1939).

⁽⁵⁾ W. RARITA and J. SCHWINGER: *Phys. Rev.*, **60**, 61 (1941).

⁽⁶⁾ S. N. GUPTA: *Phys. Rev.*, **95**, 1334 (1954).

⁽⁷⁾ P. A. MOLDAUER and K. M. CASE: *Phys. Rev.*, **102**, 279 (1956).

⁽⁸⁾ J. MATTHEWS: *Phys. Rev.*, **102**, 270 (1956).

⁽⁹⁾ The short term « Lorentz group » is used for the « Extended inhomogeneous Lorentz group », which includes translations and reflections.

⁽¹⁰⁾ E. P. WIGNER: *Ann. of Math.*, **40**, 149 (1939).

⁽¹¹⁾ L. L. FOLDY: *Phys. Rev.*, **102**, 568 (1956).

projection operator » is introduced, which greatly simplifies the treatment of the subsidiary conditions. In Sect. 4-9 the Fierz-Pauli theory for electromagnetic interaction of particles of arbitrary spin is analysed, with the aid of the spin projection operator.

In the last sections polarization operators are introduced and applied to the calculations of angular distribution of hyperon decay products.

1. - The free fields.

The simplest irreducible representation of the Lorentz group is that for which $s = 0$. The wave function is a single function of p , and satisfies the condition (a)

$$(1.1) \quad (p^2 + m^2)\varphi(p) = 0.$$

This is recognized as the Klein-Gordon equation.

Four functions $\varphi_\mu(p)$, which transform like the components of a vector, provide the basis for a reducible representation. Designating this representation by D , we have in fact

$$D = D(1) \oplus D(0),$$

where $D(s)$ is the irreducible representation corresponding to the spin value s .

Irreducible representations corresponding to any other integral spin value may be formed by taking direct products of D with itself and expanding the products in Clebsch-Gordon series. For example

$$D \otimes D = D(2) \oplus 3D(1) \oplus 2D(0).$$

The quantity that transforms according to $D^s = D \otimes D \otimes \dots \otimes D$ is the tensor of rank s . The irreducible representation $D(s)$ is $(2s+1)$ dimensional. The wave equation (1) applies to each component of the wave function, while condition (b) requires that the projections of φ on the spaces of representations of lower spin values vanish. When $s = 2$, this condition may be written

$$(1.2) \quad \begin{cases} p^{\mu_1} \varphi_{\mu_1 \mu_2} = 0, \\ \varphi_{\mu_1 \mu_2} = \varphi_{\mu_2 \mu_1}, \\ g^{\mu_1 \mu_2} \varphi_{\mu_1 \mu_2} = 0, \end{cases}$$

where $\varphi_{\mu_1 \mu_2}$ is a tensor of rank 2. Equations (2) are equivalent to Eq. (6), and

are referred to as subsidiary conditions ⁽¹²⁾. Similarly the general integral spin field is defined by the following set of equations:

$$(1.3a) \quad (p^2 + m^2)\varphi_{\mu_1 \dots \mu_s} = 0,$$

$$(1.3b) \quad \varphi_{\dots \mu_1 \dots \mu_s \dots} = \varphi_{\dots \mu_s \dots \mu_1 \dots}$$

$$(1.3c) \quad p^{\mu_1} \varphi_{\mu_1 \dots \mu_s} = 0,$$

$$(1.3d) \quad g^{\mu_1 \mu_2} \varphi_{\mu_1 \mu_2 \dots \mu_s} = 0,$$

where s is the spin.

If the spin is $s = n + \frac{1}{2}$, n integer, the wave function is a tensor of rank n , each component of which is a Dirac four spinor.

Equation (3a) is replaced by the Dirac equation ⁽¹³⁾

$$(1.3a^*) \quad (\not{p} + im)\varphi_{\mu_1 \dots \mu_n} = 0,$$

and Eq. (3d) by the subsidiary condition

$$(1.3d^*) \quad \gamma^{\mu_1} \varphi_{\mu_1 \dots \mu_n} = 0.$$

Equation (3a) is a consequence of (3a*), and the only additional information contained in (3a*) concerns the choice of reflection operators ⁽¹¹⁾.

2. - The spin projection operator.

It is convenient, temporarily, to express the subsidiary conditions (1.3b, c, d) or (1.3b, c, d*) by the symbolic notation

$$(2.1) \quad \eta_i \varphi = 0, \quad i = 1, 2, 3.$$

⁽¹²⁾ The equivalence of Eqs. (1.2) and Eq. (b) may be understood by noting that the former equates to zero all the tensors of lower rank that can be formed from $\varphi_{\mu_1 \mu_2}$. Thus, $g^{\mu_1 \mu_2} \varphi_{\mu_1 \mu_2}$ is a scalar, $p^{\mu_1} \varphi_{\mu_1 \mu_2}$ is a vector, and the antisymmetric part of $\varphi_{\mu_1 \mu_2}$ is a « six-vector », which transforms by the sum of two irreducible representations of spin 1. In Eqs. (3) below the components of $\varphi_{\mu_1} \dots$ are accounted for as follows: the 4^s components are reduced to $\binom{s+3}{3}$ by the symmetry condition. Subtracting the numbers $\binom{s+2}{3}$ and $\binom{s}{2}$ of conditions imposed by the other two subsidiary conditions leaves $2s+1$ independent components, as is appropriate for the spin 2 field.

⁽¹³⁾ Notation: $\not{p} \equiv \gamma_\mu p^\mu$, where γ_μ are the four-by-four Dirac matrices, defined by the commutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}.$$

Similarly, for (1.3a) or (1.3a*)

$$(2.2) \quad \eta\varphi = 0.$$

Introduce an orthogonal projection operator:

$$(2.3) \quad \Theta = \bar{\Theta} = \Theta^2, \quad \bar{\Theta} \equiv \gamma_4 \Theta^\dagger \gamma_4,$$

with the following properties: (i) if $\varphi_{\mu\dots}$ is a tensor of rank s (or a spin-tensor of rank $s - \frac{1}{2}$), then $(\Theta\varphi)_{\mu\dots} = \varphi_{\mu\dots}$ transforms according to $D(s)$ (i.e., $\eta_i\varphi = 0$), (ii) if a specific (spinor-) tensor $\varphi_{\mu\dots}$ transforms according to $D(s)$, then $\varphi_{\mu\dots} = (\Theta\varphi)_{\mu\dots}$, and (iii) if $\varphi_{\mu\dots}$ satisfies Eq. (2), so does $(\Theta\varphi)_{\mu\dots}$: Symbolically, omitting indices:

$$\eta_i(\Theta\varphi) \equiv 0, \quad (i),$$

$$(\eta_i\varphi = 0) \rightarrow (\varphi = \Theta\varphi), \quad (\text{Condition of uniqueness}) \quad (ii),$$

$$(\eta\varphi = 0) \rightarrow (\eta\Theta\varphi = 0), \quad (\text{Commutativity with } \eta) \quad (iii).$$

It will now be shown that the only non-trivial solution of (3) and (i) satisfies (ii) and (iii).

Uniqueness of Θ . In view of the complete symmetry of $(\Theta\varphi)_{\mu\dots}$ in all indices, Θ must be constructed from the metric tensor, the vector p_μ and (for half-odd-integer spin) γ_μ :

$$(2.4) \quad \Theta_{\mu_1\dots\mu_n}^{v_1\dots v_n} = \sum_P \{ag_{\mu_1\dots\mu_n}^{v_1\dots v_n} + bg_{\mu_1\mu_2}^{v_1v_2} \dots + \dots + d\gamma_{\mu_1}^{v_1} \gamma^{v_2\dots v_n} + \dots\},$$

where the sum is over all permutations of the lower indices. The defining equations (i) may be written

$$(2.5a) \quad \Theta_{\dots\mu_4\dots\mu_3\dots}^{v_1\dots v_n} = \Theta_{\dots\mu_3\dots\mu_4\dots}^{v_1\dots v_n},$$

$$(2.5b) \quad p^{\mu_1} \Theta_{\mu_1\dots}^{v_1\dots} = 0,$$

$$(2.5c) \quad g^{\mu_1\mu_2} \Theta_{\mu_1\mu_2\dots}^{v_1\dots} = 0, \quad 2s \text{ even},$$

$$(2.5d) \quad \gamma^{\mu_1} \Theta_{\mu_1\dots}^{\lambda_1\dots} = 0, \quad 2s \text{ odd},$$

Combination of Eqs. (4) and (5) gives

$$\Theta_{\mu_1\dots}^{v_1\dots} \Theta_{v_1\dots}^{\lambda_1\dots} = a \sum_P \Theta_{\mu_1\dots}^{\lambda_1\dots} = an! \Theta_{\mu_1\dots}^{\lambda_1\dots}.$$

Provided Θ is not identically zero, Eq. (3) yields

$$an! = 1.$$

If $\varphi_{\mu\dots}$ satisfies all the subsidiary conditions, Eq. (4) gives

$$\Theta^{\nu_1\dots} \varphi_{\nu_1\dots} = a \sum_p \varphi_{\mu_1\dots} = \varphi_{\mu_1\dots},$$

which proves that (ii) is a consequence of (3) and (i) unless Θ is identically zero. The uniqueness of Θ follows.

Commutativity of Θ with η . The consistency of Eqs. (1) with Eq. (2) requires that

$$(2.6) \quad [\eta_i, \eta]\varphi \equiv \eta'_i \varphi = 0.$$

If this condition is a consequence of Eq. (1) alone ⁽¹⁴⁾, the set (1) may be said to be complete with respect to η . If it is not, simply include the new equations (6) in the set (1), and repeat the process until a complete set of subsidiary conditions is obtained. No generality is lost therefore, by assuming that the original set was complete. This is actually true of the set (1.3b, c, d) or (1.3b, c, d*).

The commutativity of Θ with η is an immediate consequence of the completeness of the set of subsidiary conditions. The latter may be expressed by the equation

$$[\eta_i, \eta]\Theta = 0,$$

or

$$\eta_i[\Theta, \eta] = -\eta_i\eta\Theta = 0.$$

Because of (ii), this means that

$$\Theta[\Theta, \eta] = [\Theta, \eta].$$

Since Θ and either η or $i\eta$ are self-adjoint

$$[\Theta, \eta]^\dagger = \mp [\Theta, \eta].$$

Applying the last two equations

$$[\Theta, \eta] = \Theta[\Theta, \eta] = \mp \Theta[\Theta, \eta]^\dagger = \mp \Theta\{\Theta[\Theta, \eta]\}^\dagger = \mp \Theta[\Theta, \eta]^\dagger \Theta = \Theta[\Theta, \eta]\Theta = 0.$$

⁽¹⁴⁾ If, for example $\eta_1\varphi = \gamma^\mu\varphi_\mu$, $\eta_2\varphi = p^\mu\varphi_\mu$, $\eta = \mathbf{p} + im$, then $[\eta_1, \eta]\varphi \equiv \eta'_1\varphi = 2p^\mu\varphi_\mu = 2\eta_2\varphi = 0$.

This completes the proof that (iii) as well as (ii) are consequences of (i) and of Eq. (3). For integral spin (iii) is trivial; for half-odd integral spin it means that Θ commutes with \mathbf{p} .

The actual determination of Θ for arbitrary spin has been carried out in Ref. (6). The results are, for integral spin s

$$(2.7) \quad \Theta_{\alpha_1 \dots \alpha_s}^{\beta_1 \dots \beta_s} = \left(\frac{1}{s!} \right)^2 \sum_{\substack{P(\alpha) \\ P(\beta)}} \left[\prod_{i=1}^s \Theta_{\alpha_i}^{\beta_i} + a_1^{(s)} \Theta_{\alpha_1 \alpha_2} \Theta^{\beta_1 \beta_2} \prod_{i=3}^s \Theta_{\alpha_i}^{\beta_i} + \dots + \right. \\ \left. + \begin{cases} a_{(s)/2}^{(s)} \Theta_{\alpha_1 \alpha_2} \Theta^{\beta_1 \beta_2} \dots \Theta_{\alpha_{1-s} \alpha_s} \Theta^{\beta_{s-1} \beta_s} \right], & s \text{ even} \\ a_{(s-1)/2}^{(s)} \Theta_{\alpha_1 \alpha_2} \dots \Theta^{\beta_{s-2} \beta_{s-1}} \Theta_{\alpha_s}^{\beta_s} \end{cases} \right], \quad s \text{ odd},$$

where

$$a_r^{(s)} = \left(-\frac{1}{2}\right)^r s! \{r! (s-2r)! (2s-1)(2s-3) \dots (2s-2r+1)\}^{-1} \\ \Theta_{\alpha_1 \alpha_2} = g_{\alpha_1 \alpha_2} - p_{\alpha_1} p_{\alpha_2} / p^2.$$

For half-odd-integral spin $s = n + \frac{1}{2}$

$$(2.8) \quad \Theta_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_n}(s) = \frac{2s+1}{4(s+1)} \gamma^\alpha \gamma_\beta \Theta_{\alpha \alpha_1 \dots \alpha_n}^{\beta \beta_1 \dots \beta_n} (n+1).$$

This last formula provides a very simple proof of the commutativity of Θ with \mathbf{p}

$$[\Theta_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_n}(s), \mathbf{p}] = \frac{2s+1}{2(s+1)} (\gamma^\alpha p_\beta - p^\alpha \gamma_\beta) \Theta_{\alpha \alpha_1 \dots \alpha_n}^{\beta \beta_1 \dots \beta_n} (n+1) = 0.$$

One of the main uses of the spin projection is through the formulae

$$(2.9) \quad \sum \varphi_{\alpha_1 \dots} \bar{\varphi}^{\beta_1 \dots} = \Lambda^\pm \Theta_{\alpha_1 \dots}^{\beta_1 \dots},$$

for half-odd integer spin, and

$$(2.10) \quad \sum \varphi_{\alpha_1 \dots} \bar{\varphi}^{\beta_1 \dots} = \Theta_{\alpha_1 \dots}^{\beta_1 \dots},$$

for integer spin, where the sums are over the positive energy solutions of the wave equation and the subsidiary conditions. Equation (2.9) has been used in Sect. 10 and in reference (6).

3. - Electromagnetic interaction ⁽¹⁶⁾.

The first attempt to introduce an interaction between the higher spin field and the Maxwell field was made by DIRAC ⁽¹⁷⁾, who started from Eq. (1.3)

⁽¹⁵⁾ R. E. BEHRENDs and C. FRONSDAL: *Phys. Rev.*, **106**, 345 (1957).

⁽¹⁶⁾ Only half-odd integer spins are considered in Sect. 3-6. Summary results for integer spin are given in Sect. 7.

⁽¹⁷⁾ P. A. M. DIRAC: *Proc. Roy. Soc., A* **155**, 447 (1936).

and carried out the substitution

$$(3.1) \quad p_\mu \rightarrow p_\mu - ieA_\mu.$$

Although this substitution represents the only known method of introducing a gauge-invariant direct interaction between A_μ and the particle field, it does not uniquely determine the form of the interaction. Indeed, different theories are obtained by starting from different, though equivalent, formulations of the free field theory.

It was shown by FIERZ and PAULI ⁽⁴⁾ that Dirac's method leads to inconsistent equations for the interaction of φ with an arbitrary external electromagnetic field. An alternative method was proposed, that avoids the inconsistency of the Dirac theory, without completely removing the ambiguity.

Although it is not certain that the Fierz-Pauli theory is self-consistent (in particular difficulties seem to present themselves with regard to quantization ⁽¹⁸⁾), it has been rather widely accepted. It is, therefore, of some interest to exhibit, in a simple way, the intimate relationship between the various formulations of the Fierz-Pauli theory, as well as the importance of the spin projection operator in this connection.

The theory of Fierz and Pauli is based on the requirement that all the field equations be derivable from a single Lagrangian variational principle ⁽¹⁹⁾. While this requirement is satisfied by any formulation of the free field theory, the same may not hold after the substitution (1) has been carried out. The method therefore consists of determining a Lagrangian for the free fields, and then carrying out the substitution in the Lagrangian.

Application of Eq. (2.5) shows that the wave equation and subsidiary condition (1.3) may be deduced very simply from the following equation

$$(3.2) \quad (\Theta \mathbf{p} + im)\varphi = 0.$$

In the momentum representation p_μ is a c -number, so that Eq. (4.2) may be obtained from the Lagrangian

$$(3.3) \quad \mathcal{L} = \bar{\varphi}(\Theta \mathbf{p} + im)\varphi$$

by variation with respect to φ . However, since $\Theta \mathbf{p}$ involves inverse powers of p^2 , it does not seem possible to treat with conventional methods the theory

⁽¹⁸⁾ S. KUSAKA and J. W. WEINBERG: *University of California Doctoral Dissertation* of J. W. WEINBERG (Berkeley, 1940).

⁽¹⁹⁾ That is, all equations which are postulated «a priori» (i.e., before the variation of the Lagrangian is carried out.) «A priori» conditions on the wave function are permissible provided there are properly taken into account under the variation: the variations $\delta\varphi_{\mu_1} \dots$ are not all independent. This was overlooked in the theory of Møller and Case.

which would result from applying (1) to Eq. (2). Indeed, it is a practical necessity that the Lagrangian be linear in p_μ .

The remainder of this section deals with the problem of replacing (3) by a Lagrangian which is linear in p_μ , with the objective of relating to each other the various existing formulations of the Fierz-Pauli theory. The discussion is limited to the case of spin $\frac{3}{2}$, since much of the literature treats this case exclusively. In the following section a more general approach is taken.

The case of spin $\frac{3}{2}$. Eq. (2) may be reduced to the form of a first-order differential equation. The corresponding first order Lagrangian is equivalent to (3) in the sense that variation leads to the same equations (wave equation as well as subsidiary conditions) for φ_μ . After the substitution (1) has been carried out, this will no longer be true.

Eq. (2) may be written, omitting indices

$$(3.4) \quad -im\varphi = (1 - \frac{1}{4}\gamma\gamma)\mathbf{p}\Theta\varphi',$$

where φ' is the projection of φ which satisfies $\gamma\cdot\varphi = 0$:

$$(3.5) \quad \varphi' \equiv (1 - \frac{1}{4}\gamma\gamma)\varphi.$$

The explicit expression for Θ is

$$(3.6) \quad \Theta = 1 - \frac{1}{3}\gamma\gamma - \frac{1}{3p^2}(\mathbf{p}\gamma\mathbf{p} + \mathbf{p}\gamma\mathbf{p}).$$

When Eq. (6) is inserted into Eq. (4), the result may be written

$$(3.7) \quad -im\varphi = (1 - \frac{1}{4}\gamma\gamma)(\mathbf{p}\varphi' - \mathbf{p}\psi),$$

where

$$(3.8) \quad \psi \equiv \frac{2}{3p^2}\mathbf{p}\mathbf{p}\cdot\varphi$$

is a four-component spinor field.

Multiplying Eq. (8) by $-im$, and substituting for $-im\varphi$ from Eq. (7), there results

$$(3.9) \quad -im\psi = -\frac{1}{2}\mathbf{p}\psi + \frac{1}{3}\mathbf{p}\cdot\varphi'.$$

Eqs. (7), (9) are equivalent to the original set (1.3), and are seen to be identical to the equations of Fierz and Pauli, in the form obtained in the Appendix.

Later formulations of the Fierz-Pauli theory do not require the use of the auxiliary field ψ . In the following it will be seen how ψ can be eliminated, and a general theory obtained that includes, as special cases, the theories of Rarita and Schwinger ⁽⁵⁾, Harish-Chandra ⁽²⁰⁾ and Moldauer and Case ⁽⁷⁾.

Equation (7) is equivalent to

$$(3.10) \quad -im\varphi' = (1 - \tfrac{1}{4}\gamma\gamma)(p\varphi' - p\psi),$$

$$(3.11) \quad \gamma \cdot \varphi = 0.$$

The vanishing of $p \cdot \varphi'$ and hence of ψ does not depend on Eq. (11), but may be inferred from Eqs. (9), (10). Since the latter do not involve the projection

$$\varphi'' = \tfrac{1}{4}\gamma\gamma \cdot \varphi,$$

which is orthogonal to φ' , it is possible to identify the field ψ (which vanishes by virtue of Eqs. (9), (10)), with the field $\gamma \cdot \varphi = \gamma \cdot \varphi''$ (which vanishes by virtue of Eq. (11)). Thus the auxiliary field ψ may be eliminated by writing

$$(3.12) \quad \psi = a\gamma \cdot \varphi, \quad a \neq 0.$$

This makes Eq. (11) superfluous, while Eqs. (9), (10) become, respectively

$$(3.13) \quad -im\varphi'' = -\frac{1}{8}\gamma p\gamma \cdot \varphi'' + \frac{1}{12a}\gamma p \cdot \varphi',$$

$$(3.14) \quad -im\varphi' = (1 - \tfrac{1}{4}\gamma\gamma)(p\varphi' - ap\gamma \cdot \varphi'').$$

Eqs. (13), (14) may be replaced by any linear combination of them

$$(3.15) \quad k(3.13) + (3.14), \quad k \neq 0,$$

as is seen by multiplying first by $\tfrac{1}{4}\gamma\gamma$ and then by $(1 - \tfrac{1}{4}\gamma\gamma)$

$$\tfrac{1}{4}\gamma\gamma(3.15) = k(3.13),$$

$$(1 - \tfrac{1}{4}\gamma\gamma)(3.15) = (3.14).$$

⁽²⁰⁾ HARSH-CHANDRA: *Phys. Rev.*, **71**, 793 (1947). The theories of References ⁽⁵⁾ and ⁽⁷⁾, as well as the part of Reference ⁽²⁰⁾ which is of interest in the present connection, are all equivalent to the Fierz-Pauli theory. The latter, however, suffers from a cumbersome formulation (see Appendix).

When Eq. (15) is written out in full, and multiplied by $\bar{\varphi}$, the Lagrangian

$$(3.16) \quad \mathcal{L} = \bar{\varphi} \left\{ im \left(1 - \frac{1}{4} (1 - k) \gamma\gamma \right) + \mathbf{p} - \left(\frac{1}{2} + a \right) \mathbf{p}\gamma + \right. \\ \left. + \left(-\frac{1}{2} + \frac{k}{12a} \right) \gamma\mathbf{p} + \left(\frac{3}{8} + \frac{a}{4} - \frac{k}{8} - \frac{k}{48a} \right) \gamma\mathbf{p}\gamma \right\} \varphi$$

is obtained.

In order that the equation derived from (16) by variation of φ be the adjoint of Eq. (15), the Lagrangian must be Hermitian. This imposes the condition

$$(3.17) \quad k = -12aa^* \neq 0,$$

and (16) reduces to, with $A = -a - \frac{1}{2} \neq -\frac{1}{2}$,

$$(3.18) \quad \mathcal{L} = \bar{\varphi} \{ (\mathbf{p} + im) + A\mathbf{p}\gamma + A^*\gamma\mathbf{p} + \frac{1}{2}(3AA^* + A + A^* + 1)\gamma\mathbf{p}\gamma - \\ - im(3AA^* + \frac{3}{2}A + \frac{3}{2}A^* + 1)\gamma\gamma \} \varphi.$$

This is a two-parameter family of Lagrangians, and reduces to a one-parameter family which is identical to that obtained by MOLDAUER and CASE⁽⁷⁾, when A is taken to be real. The Rarita-Schwinger⁽⁵⁾ theory is obtained by the special choice $A = -\frac{1}{3}$. From the theory of Harish-Chandra⁽²⁰⁾, which describes particles of mixed spins, $\frac{1}{2}$ and $\frac{3}{2}$, PETRAS⁽²¹⁾ has extracted the spin $\frac{3}{2}$ part. Petras' Lagrangian, which is not Hermitian, corresponds to the special choice $k = 1$, $a = -\frac{1}{2} \pm \sqrt{\frac{1}{3}}$.

The complete generality of (18) is not guaranteed by the above analysis, but is proved rigorously in Sect. 5. The proof rests on some general theorems concerning first order wave equations for arbitrary spin; these are obtained in Sect. 4 for half-odd integer spin, and in Sect. 7 for integer spin.

4. - The minimal condition.

The most general Euler-Lagrange equation which is obtained from a Hermitian; Lorentz invariant Lagrangian, and which is of the first order in p_μ , and linear in φ , may be written

$$(4.1) \quad (p^\mu \alpha_\mu + im\beta)\varphi = 0, \quad \alpha_\mu^+ = \alpha_\mu, \quad \beta^+ = \beta,$$

where the matrices α_μ and β are form-invariant, numerical matrices⁽²²⁾.

⁽²¹⁾ M. PETRAS: *Czech. Journ. Phys.*, **5**, 2 (1955).

⁽²²⁾ Note that in the spin $\frac{1}{2}$ case, $\alpha_\mu = \gamma_\mu$, $\beta = 1$.

The wave function φ transforms according to some reducible representation D_φ of the Lorentz group. The transformations consist of an «orbital part» and a «spin part», and the two parts commute. Therefore the spin matrices are, by themselves, a representation of the Lorentz group; $D_\varphi^{(s)}(L)$ say. This may be written as a direct sum of irreducible representations $D(i, L)$, with $i = \frac{1}{2}, \frac{3}{2}, \dots, s$, (we are dealing in this section with half-odd-integer spin only)

$$(4.2) \quad \begin{cases} D_\varphi^{(s)}(L) = D(s, L) \oplus D(s-1, L) \oplus \dots \oplus D(s-1, L) \oplus D(s-2, L) \oplus \dots \\ \quad = D(s, L) \oplus \sum_{i=\frac{1}{2}}^{s-1} n_i D(i, L), \end{cases}$$

The spin value s appears once, while the number n_i is the multiplicity of the spin value i .

Commutation relations for the matrices α_μ and β will now be derived from the following requirements. First, that Eq. (1) be equivalent to the following two equations

$$(4.3) \quad (1 - \Theta)\varphi = 0,$$

$$(4.4) \quad (\mathbf{p} + im)\Theta\varphi = 0.$$

Here $\Theta\varphi$ is the part of φ which transforms according to the first term of Eq. (2). Thus Eq. (3) is the subsidiary condition which requires that the spin be unique, and Eq. (4) is the wave-equation which we impose on $\Theta\varphi$ only. Notice that the representation is not necessarily in terms of Dirac spinors, so that Eq. (4) means that there is a wave equation which would take that form if such a representation were used.

Eq. (4) is the condition that the mass be unique, but reflects also the choice of reflection operators. Specifically, it is invariant under the substitution

$$(4.5) \quad \Theta\varphi(p_4, \mathbf{p}) \rightarrow \gamma_4 \Theta\varphi(p_4, -\mathbf{p}).$$

The reason for demanding this invariance is simply the success of the Dirac equation for spin $\frac{1}{2}$. As long as one is discussing free fields only, there is no reason to extend invariance under (5) to hold for the entire wave-function. However, when the electromagnetic interaction is introduced, Eq. (3) will no longer hold. Analogy with the spin $-\frac{1}{2}$ case therefore suggests that Eq. (1) be invariant under a transformation of the form

$$(4.6) \quad \varphi(p_4, \mathbf{p}) \rightarrow \gamma_4 \varphi(p_4, -\mathbf{p}).$$

The choice of γ_4 in Eq. (6) is not imperative, but any other choice consistent

with Eq. (5) is easily seen to be equivalent for our purpose. An alternative way of stating the invariance under (6) is to require that, if a representation in terms of 4-spinors is used, the α_μ and β be expressible in terms of γ_μ -matrices and the metric tensor (i.e. no σ -matrices or γ_5 will appear).

Thus, the requirements from which commutation relations for α_μ and β will be derived are the equivalence of Eq. (1) to the set (3), (4), plus the invariance of Eq. (1) under (6).

A component of φ may be labelled by the number i , referring to the representation $D(i, L)$ (see Eq. (2)), a number t_i , ($1 \leq t_i \leq n_i$) which refers to one of the n_i representations $D(i, L)$, a number s_i taking on $2i+1$ values and a sign (the sign of the energy). The two latter refer to the $2(2i+1)$ components of $D(i, L)$. Hence

$$(4.7) \quad \varphi = |t_i, i, s_i, \varepsilon\rangle, \quad \varepsilon = \pm.$$

It is clear that, by definition

$$(4.8) \quad \langle \varepsilon, s_i, i, t_i | \Theta | t'_i, i', s'_i, \varepsilon' \rangle = \delta_{i,s} \delta_{i',s} \delta_{t_i t'_i} \delta_{s_i s'_i} \delta_{\varepsilon \varepsilon'}.$$

Consider the sub-group L^0 of L under which a given momentum vector p^0 is invariant. This was called the little group by WIGNER⁽¹⁰⁾, who proved that $D_\varphi^{(s)}(L^0)$ may be written as the following direct sum of irreducible representations

$$(4.9) \quad D_\varphi^{(s)}(L^0) = D^+(s, L^0) \oplus D^-(s, L^0) \oplus \sum_{i=\frac{1}{2}}^{s-\frac{1}{2}} n_i [D^+(i, L^0) \oplus D^-(i, L^0)].$$

Form-invariance of α_μ and β under L , means that $\alpha_\mu p^{0\mu}$ and β commute with $D_\varphi^{(s)}(L^0)$. By Schur's lemma:

$$(4.10) \quad \langle \varepsilon, s_i, i, t_i | \alpha_\mu p^{0\mu} | t'_i, i', s'_i, \varepsilon' \rangle = \delta_{i,i'} \delta_{s_i, s'_i} \langle \varepsilon, s_i, i, t_i | \alpha_\mu p^{0\mu} | t'_i, i, s_i, \varepsilon' \rangle$$

and similarly for β . Invariance under (6) gives (as is easily seen in the rest-system of $p^{0\mu}$)

$$\begin{aligned} \langle \varepsilon, s_i, i, t_i | \alpha_\mu p^{0\mu} | t'_i, i, s_i, \varepsilon' \rangle &= (\mathbf{p}^0)_{\varepsilon \varepsilon'} a(i)_{t_i t'_i}, \\ \langle \varepsilon, s_i, i, t_i | \beta | t'_i, i, s_i, \varepsilon' \rangle &= \delta_{\varepsilon \varepsilon'} b(i)_{t_i t'_i}. \end{aligned}$$

Formally

$$(4.11) \quad \alpha_\mu p^\mu = \mathbf{p} \otimes [a(s) \oplus \sum_{i=\frac{1}{2}}^{s-\frac{1}{2}} a(i)_{t_i t'_i}],$$

$$(4.12) \quad \beta = [b(s) \oplus \sum_{i=\frac{1}{2}}^{s-\frac{1}{2}} b(i)_{t_i t'_i}],$$

where $a(i)_{i_i i'_i}$ and $b(i)_{i_i i'_i}$ are numerical matrices diagonal in the dimension i . Here the superscript «0» on $p^{0\mu}$ has been dropped, since Eqs. (11) and (12) are explicitly covariant.

Comparison of Eqs. (8), (11) and (12) gives immediately (since $a(s)$ and $b(s)$ are one-dimensional)

$$(4.13) \quad \alpha_\mu p^\mu \Theta = \Theta \alpha_\mu p^\mu = \mathbf{p} \Theta a(s),$$

$$(4.14) \quad \beta \Theta = \Theta \beta = \Theta b(s).$$

Hence Eq. (11) may be written as two separate equations

$$(4.15) \quad [\mathbf{p} a(s) + im b(s)] \Theta \varphi = 0,$$

$$(4.16) \quad [\alpha_\mu p^\mu + im \beta](1 - \Theta) \varphi = 0.$$

Up to this point the form-invariance of α_μ and β plus invariance of Eq. (1) under (6) have been exploited. There remains the requirement that Eq. (1) be equivalent to Eqs. (3) and (4). In view of the above this means, first, that

$$(4.17) \quad a(s) = b(s) = 1,$$

and, second, that Eq. (16) reduce to Eq. (3), i.e. that the secular determinant

$$(4.18) \quad \text{Det}[(\alpha_\mu p^\mu + im \beta)(1 - \Theta)]$$

must be non-zero for all values of m . It is seen that this is impossible for $m = 0$, so that this case must be excepted. Next, β must be non-singular, or more precisely, $\beta \varphi$ must have the same number of components as φ . Then there exists a matrix defined by

$$\beta^{-1} \beta \varphi = \varphi.$$

The determinant (4.18) may then be replaced by

$$(4.19) \quad \text{Det}[(\Gamma_\mu p^\mu + im)(1 - \Theta)], \quad \Gamma_\mu = \beta^{-1} \alpha_\mu.$$

According to Eqs. (11) and (12),

$$(4.20) \quad \Gamma_\mu p^\mu = \mathbf{p} \otimes [1 \oplus \sum_{i=\frac{1}{2}}^{s-1} C(i)_{i_i i'_i}], \quad C(i) = b^{-1}(i) a(i),$$

and (4.18) is the product of the determinants

$$(4.21) \quad \text{Det}[\mathbf{p} \otimes C(i)_{i_i i'_i} + im \delta_{i_i i'_i}], \quad i = s, s-1, \dots, \frac{1}{2}.$$

These determinants are non-zero only if they are all of the form

$$(im)^{n_i}.$$

But every matrix satisfies its characteristic equation, hence

$$(4.22) \quad [C(i)]^{n_i} = 0.$$

This is the main result. It may be written in a number of ways, the most striking being Eq. (27) below. Conditions of this form are referred to as minimal conditions.

It has been proved that the necessary and sufficient conditions that Eq. (1), with form invariant α_μ and β , be equivalent to Eqs. (3) and (4), and be invariant under (6), are that the matrices satisfy the conditions

$$(4.23) \quad \Gamma_\mu p^\mu \Theta = \Theta \Gamma_\mu p^\mu = p \Theta,$$

$$(4.24) \quad \beta \Theta = \Theta \beta = \Theta,$$

$$(4.25) \quad \Gamma_\mu p^\mu = p \otimes [1 \oplus \sum_{i=\frac{1}{2}}^{s-1} C(i)_{i_i i'_i}],$$

$$(4.26) \quad \beta = 1 \oplus \sum_{i=\frac{1}{2}}^{s-1} b(i)_{i_i i'_i},$$

$$(4.27) \quad (\Gamma_\mu p^\mu)^{\bar{n}} \Theta = (p)^{\bar{n}} \Theta,$$

where

$$(4.28) \quad \Gamma_\mu = \beta^{-1} \alpha_\mu,$$

and \bar{n} is the largest of the numbers n_i . Eqs. (23) and (24) are simple restatements of Eqs. (13), (14) and (17). Eq. (27) is completely equivalent to Eq. (22).

Note. — If the wave-function is a tensor spinor of rank $s - \frac{1}{2}$, and symmetric in all the tensor indices, $\bar{n} = n_{\frac{1}{2}} = s + \frac{1}{2}$, so that

$$(\Gamma_\mu p^\mu)^{s+\frac{1}{2}} = (p)^{s+\frac{1}{2}} \Theta.$$

This can be realized only if $s = \frac{1}{2}$ or $\frac{3}{2}$, since only then is the right-hand side a polynomial in p_μ . This proves a statement by KUSAKA and WEINBERG⁽¹⁸⁾, that a symmetric tensor-spinor of rank 2 is insufficient to describe the spin $\frac{5}{2}$ field by a first order wave equation. If, however, a general (i.e., not symmetric) spinor-tensor is used, \bar{n} will always satisfy the minimum requirement of making the right-hand side of Eq. (27) a polynomial in p_μ .

5. - Wave equations for spin $\frac{3}{2}$

In Sect. 3 it was seen that the spinor-tensor φ_μ is adequate for the construction of a first-order wave equation for spin $\frac{3}{2}$, and a Lagrangian (Eq. (3.18)) was constructed. With the aid of the results of the preceding section, the complete generality of that Lagrangian will be demonstrated.

The reduction of φ_μ according to irreducible representations of the Lorentz group may be carried out as follows

$$\varphi_\mu = \begin{bmatrix} \Theta_\mu{}^\nu & \varphi_\nu \\ A^\nu & \varphi_\nu \\ B^\nu & \varphi_\nu \end{bmatrix},$$

with

$$A^\nu \varphi_\nu = \sqrt{(1/12)} (\gamma^\nu - 4p^\nu p/p^2) \varphi_\nu,$$

$$B^\nu \varphi_\nu = \frac{1}{2} \gamma^\nu \varphi_\nu.$$

Here $\Theta\varphi$ is the spin $\frac{3}{2}$ part, having 8 independent components, and the other two parts have each the four components appropriate to spin $\frac{1}{2}$ fields. Furthermore

$$A \cdot B = B \cdot A = 0,$$

$$A \cdot A = B \cdot B = 1.$$

In this representation the explicit form of $\Gamma_\lambda p^\lambda$, as given by (4.25) is

$$(5.1) \quad (\Gamma_\lambda p^\lambda) = \Theta p + AC_{11} pA + AC_{12} pB + BC_{21} pA + BC_{22} pB.$$

It is now a simple matter to impose the requirements 1), that $\Gamma_\lambda p^\lambda$ be of the first order in p_μ , and 2), that the square of the matrix C_{ij} be zero, as required by Eq. (4.27). Next the most general form of the matrix β is written down

$$(5.2) \quad \beta = 1 + \text{const } \gamma\gamma.$$

It is immediately found that the Lagrangian

$$\mathcal{L} = \bar{\varphi} \{ \beta \Gamma_\lambda p^\lambda + im\beta \} \varphi,$$

with $\Gamma_\lambda p^\lambda$ and β given by (5.1) and (5.2), is identical to the one found in

Sect. 3 (Eq. 3-16). This constitutes a rigorous proof that the latter is indeed the most general Lagrangian which is linear in p_μ , and from which all the field equations may be derived by a single variation, when the wave equation for the spin $\frac{3}{2}$ field is taken to be the tensor-spinor φ_μ . In particular (3.18) is the most general hermitian Lagrangian, under these conditions.

6. - Algebra of the Γ -matrices.

Commutation-relations may be derived for the matrices $\Gamma_\mu = \beta^{-1}\alpha_\mu$ from the minimal condition (4.27) and from (4.23), (4.24). In terms of Γ_μ

$$(6.1) \quad \left\{ \begin{array}{l} (\Gamma_\mu p^\mu)^{\bar{n}} = \mathbf{p}^{\bar{n}} \Theta, \\ (\Gamma_\mu p^\mu) \Theta = \Theta (\Gamma_\mu p^\mu) = \mathbf{p} \Theta, \end{array} \right.$$

or

$$(6.2) \quad (\Gamma_\mu p^\mu - \mathbf{p})(\Gamma_\nu p^\nu)^{\bar{n}} = 0.$$

In the rest system

$$(6.3) \quad (\Gamma_4 - \gamma_4) \Gamma_4^{\bar{n}} = 0.$$

In these equations \bar{n} is the largest number of fields of given spin that appear in the wave-function φ . The existence of a relation of the form of Eq. (3) was derived by KUSAKA and WEINBERG⁽¹⁸⁾, who did not give the present definition of the number \bar{n} . Neither was Eq. (1) given by these authors. The weaker condition

$$(6.4) \quad (\Gamma_4^2 - 1) \Gamma_4^{\bar{n}} = 0,$$

has been given by HARISH-CHANDRA⁽²⁰⁾ and by UMEZAWA and VISCONTI⁽²²⁾. Although the methods of these authors differ somewhat from each other, the basis of the argument is in each case the requirement that every component of the wave-function satisfy the Klein-Gordon equation. In the present analysis Eq. (4), as well as the stronger condition (2) have been derived from the requirements that the solutions of Eq. (1) describe particles of unique spin, and that a unitary parity operator exist.

In the rest system $\Theta = \Gamma_4^{\bar{n}}$, by Eq. (1). Hence $\Gamma_4^{\bar{n}}$ is an idempotent. This can also be seen by iteration of Eq. (4).

Since the p_μ are arbitrary, conditions more general than Eq. (3) may be deduced from Eq. (2)

$$(6.5) \quad \sum_{P(\mu)} (\Gamma_\mu - \gamma_\mu) \Gamma_{\mu_1} \dots \Gamma_{\mu_{\bar{n}}} = \sum_{P(\mu)} \Gamma_{\mu_1} \dots \Gamma_{\mu_{\bar{n}}} (\Gamma_\mu - \gamma_\mu) = 0,$$

and the weaker conditions

$$(6.6) \quad \sum_{P(\mu)} (\Gamma_\mu \Gamma_{\mu'} - \delta_{\mu\mu'}) \Gamma_{\mu_1} \dots \Gamma_{\mu_{\bar{n}}} = \sum_{P(\mu)} \Gamma_{\mu_1} \dots \Gamma_{\mu_{\bar{n}}} (\Gamma_\mu \Gamma_{\mu'} - \delta_{\mu\mu'}) = 0,$$

where the sums are over all permutations of the indices. Special examples of Eqs. (5), (6) are needed for the calculation in Sect. 8 of magnetic moments

$$(6.7) \quad \Gamma_4^{\bar{n}} (\Gamma_\mu - \gamma_\mu) \Gamma_4^{\bar{n}} = 0,$$

$$(6.8) \quad P_+ [\Gamma_i (\sum_{\bar{i}=0}^{\bar{n}} \Gamma_4^{\bar{i}} - \frac{1}{2} \Gamma_4^{\bar{n}}) \Gamma_j + \Gamma_j (\sum_{\bar{i}=0}^{\bar{n}} \Gamma_4^{\bar{i}} - \frac{1}{2} \Gamma_4^{\bar{n}}) \Gamma_i] P_+ = \delta_{ij},$$

$$(6.9) \quad \Gamma_4 P_\pm = \pm P_\pm,$$

where $i, j = 1, 2, 3$ and

$$(6.10) \quad P_\pm = \frac{1}{2} (1 \pm \Gamma_4) \Gamma_4^{\bar{n}} = \frac{1}{2} \Gamma_4^{\bar{n}} (1 \pm \Gamma_4).$$

7. - Integer spin fields.

In the case of integer spin, not first-order but second order wave equations are considered

$$(7.1) \quad (\alpha_{\mu\nu} p^\mu p^\nu + m^2 \beta) \varphi = 0.$$

In the simplest non-trivial case, that of spin 2, it proves sufficient to work with a symmetric, traceless second rank tensor $\varphi_{\mu\nu}$, and one auxiliary scalar field ψ . Starting from the wave equation (in terms of an arbitrary 2-nd rank tensor)

$$(\Theta p^2 + m^2) \varphi = 0,$$

a field φ' that satisfies the algebraic subsidiary conditions, is introduced

$$(7.2) \quad \varphi'_{\mu\nu} = [\frac{1}{2} (\delta_\mu^\lambda \delta_\nu^\rho + \delta_\mu^\rho \delta_\nu^\lambda) - \frac{1}{4} \delta_{\mu\nu} \delta^{\lambda\rho}] \varphi_{\lambda\rho}.$$

The terms in Θp^2 containing powers of $1/p^2$ are absorbed into a new scalar field ψ , which is subsequently identified with φ_μ^μ : There result two equations that may be added in analogy with Eqs. (3.13), (3.14), to give a 2-parameter family of Lagrangians.

Considerations very analogous to those of Sect. 4 may be applied to the integer spin case. The main results of Sect. 4 are contained in Eqs. (4.23-27),

and have the following analogues, respectively

$$(7.3) \quad (p \cdot \alpha \cdot p) \Theta = \Theta (p \cdot \alpha \cdot p) = p^2 \Theta,$$

$$(7.4) \quad \beta \Theta = \Theta \beta = \Theta,$$

$$(7.5) \quad p \cdot \Gamma \cdot p = p^2 [1 \oplus \sum_{i=0}^{s-1} C(i)_{t_i t'_i}]$$

$$(7.6) \quad \beta = [1 \oplus \sum_{i=0}^{s-1} b(i)_{t_i t'_i}],$$

$$(7.7) \quad (p \cdot \Gamma \cdot p)^{\bar{n}} = p^{2\bar{n}} \Theta,$$

where

$$(7.8) \quad \Gamma_{\mu\nu} \equiv \beta^{-1} \alpha_{\mu\nu},$$

and β^{-1} is defined by

$$\beta^{-1} \beta \varphi = \varphi.$$

These results may be applied to the derivation of the most general wave equation for spin 2, in terms of the wave function $\varphi_{\mu\nu}$. For a traceless, symmetric $\varphi_{\mu\nu}$, the dimensions of the $C(i)$ matrices are $n_1 = n_0 = 1$. Thus $\bar{n} = 1$, and Eq. (7) cannot be satisfied. Relaxing the trace condition on $\varphi_{\mu\nu}$, there results $n_1 = 1$, $n_0 = \bar{n} = 2$. Alternatively, the symmetry condition may be given up, so that $n_1 = \bar{n} = 3$, $n_0 = 1$.

In the general case of integer spin s , the wave function may be taken to be a tensor of rank s . A completely symmetric tensor cannot be used, since then $\bar{n} = n_0 = \frac{1}{2}(s+2)$ or $\frac{1}{2}(s+1)$, according to whether s is even or odd, respectively. But this does not make the right hand side of Eq. (5) a polynomial, except when $s = 2$. If, instead $\varphi_{\mu\nu\dots}$ is completely traceless, $n_{s-i} = \binom{s}{i}$, which in general is much larger than what is required by Eq. (5).

From Eqs. (3), (4), (5) commutation relations may be derived in the form

$$\sum_{P(\mu)} \Gamma_{\mu_1 \mu_2} \dots \Gamma_{\mu_{\bar{n}-1} \mu_n} (\Gamma_{\mu \mu'} - g_{\mu \mu'}) = 0.$$

8. - Magnetic moment of fermions.

The magnetic moment of particles defined by an equation of the form

$$(8.1) \quad [\Gamma_\mu \Pi^\mu + im] \varphi = 0, \quad \Pi^\mu = p^\mu - ieA^\mu,$$

was considered by HARISH-CHANDRA ⁽²⁴⁾. HARISH-CHANDRA finds, in the non-

⁽²³⁾ H. UMEZAWA and A. VISCONTI, see H. UMEZAWA: *Quantum Field Theory* (Amsterdam 1956).

⁽²⁴⁾ HARISH-CHANDRA: *Proc. Roy. Soc., A* **195**, 195 (1948).

relativistic approximation, that Eq. (1) reduces to

$$(8.2) \quad \left\{ (H^4 + im) - \frac{1}{2im} H^i H^j P_+ \Gamma_i (P_- + 2 \sum_{i=0}^{\bar{n}-1} \Gamma_4^i P_0) \Gamma_j \right\} P_+ \varphi = 0,$$

where P_{\pm} were defined by Eq. (6.10) and

$$P_0 = 1 - \Gamma_4^{\bar{n}}.$$

This result relies on Eq. (6.4) only, and the number \bar{n} remains unspecified in Harish-Chandra's theory.

Noting that the form invariance of Γ_{μ} requires that

$$(8.3) \quad \Gamma_i = \Gamma_4 S_{4i} - S_{4i} \Gamma_4,$$

where S_{4i} are the spin transformation matrices, there follows by a generalization of a technique due to Petras

$$(8.4) \quad P_+ \Gamma_i (P_- + 2 \sum_{i=0}^{\bar{n}-1} \Gamma_4^i P_0) \Gamma_j P_+ = \frac{1}{2} P_+ (\Gamma_4 S_{4i} - S_{4i} \Gamma_4) (P_- + 2 \sum_{j=0}^{\bar{n}-1} \Gamma_4^j P_0) \Gamma_j P_+ + \\ + \frac{1}{2} P_+ \Gamma_i (P_- + 2 \sum_{i=0}^{\bar{n}-1} \Gamma_4^i P_0) (\Gamma_4 S_{4j} - S_{4j} \Gamma_4) P_+ = P_+ (S_{4i} \Gamma_j - \Gamma_j S_{4j}) P_+.$$

When this result is introduced into Eq. (2), that equation becomes

$$\left\{ (H^4 + im) - \frac{1}{2im} H^i H^j - F^{ij} M_{ij} \right\} P_+ \varphi = 0,$$

where the magnetic moment M_{ij} is given by

$$(8.5) \quad M_{ij} = \frac{e}{2m} P_+ (S_{4i} \Gamma_j - S_{4j} \Gamma_i) P_+.$$

Any relativistic wave function may be considered as being composed of two parts as follows. One part contains the spin s field, and is a tensor-spinor of rank $n = s - \frac{1}{2}$. This part of the complete wave function φ (written without indices) shall be written $\varphi_{\mu_1 \dots \mu_n}$ (with indices). The other part of φ does not contain a spin s field, and must therefore vanish as a consequence of the field equations. This part will be written ψ . The projection $(\Theta\varphi)$ is the set of wave

functions which satisfy the following conditions

- (i) $p^{\mu_1} \varphi_{\mu_2 \dots \mu_n} = 0,$
- (ii) $\gamma^{\mu_1} \varphi_{\mu_1 \dots \mu_n} = 0,$
- (iii) $\varphi_{\dots \mu_1 \dots \mu_j \dots} = \varphi_{\dots \mu_j \dots \mu_1 \dots},$
- (iv) $\psi = 0.$

Let an infinitesimal Lorentz transformation be carried out

$$(8.6) \quad \begin{aligned} p^\mu &\rightarrow p'^\mu, \\ \varphi(p) &\rightarrow \varphi(p') + d\omega S_{4i} \varphi(p). \end{aligned}$$

When these substitutions are made in Eqs. (i)-(iv), there follows that

$$(\eta_r \varphi = 0) \rightarrow (\eta_r S_{4i} \varphi = 0).$$

Here $\eta_r \varphi = 0$, $r = 1, 2, 3$ stands for the three conditions (ii), (iii) and (iv). By the definition of Θ this may be written

$$\eta_r S_{4i} \Theta = 0,$$

or, in the rest system

$$(8.7) \quad \eta_r S_{4i} P_+ = 0.$$

With respect to (iv), this means that the part ψ of the wave function may be ignored in the calculation of the magnetic moment. Then the Γ_i -matrices must be expressed in terms of the invariant matrices $g_{\alpha\beta}$, γ_α , and $\varepsilon_{\alpha\beta\gamma\delta}$. When an expression of this form is introduced into (5), it is immediately recognized that, because of Eq. (7), the only surviving term is that which is diagonal in all tensor indices. By Eq. (6.7), this term is equal to γ_i . Hence

$$(8.8) \quad M_{ij} = (e/2m) P_+ (S_{4i} \gamma_j - S_{4j} \gamma_i) P_+.$$

The explicit form of S_{4i} is (ignoring the part which acts on ψ)

$$(S_{4i})_{\alpha_1 \dots}^{\beta_1 \dots} = \frac{1}{2} \gamma_i \gamma_\alpha \delta_{\alpha_1 \dots}^{\beta_1 \dots} + \sum_m \delta_{\alpha_1 \dots}^{\beta_1 \dots} (\delta_{4\alpha_m} \delta_i^{\beta_m} - \delta_{i\alpha_m} \delta_4^{\beta_m}) \delta_{\alpha_{m+1} \dots}^{\beta_{m+1} \dots}$$

When this is inserted into (8), all but the first term is annihilated by the projection operator P_+ , and there remains

$$M_{ij} = (e/2m) P_+ \gamma_i \gamma_j P_+.$$

By means of some simple algebra

$$\begin{aligned}
 (P_+)^{\beta_1 \dots \gamma_i \gamma_j (P_+)^{\gamma_1 \dots} &= \\
 &= (P_+)^{\beta_1 \dots (\delta_{\beta_1}^{\delta_1} - \frac{1}{2}(1 - 1/2s)\gamma_{\beta_1} \gamma^{\delta_1}) \gamma_i \gamma_j (\delta_{\delta_1}^{\epsilon_1} - \frac{1}{2}(1 - 1/2s)\gamma_{\delta_1} \gamma^{\epsilon_1}) (P_+)^{\gamma_1 \dots} = \\
 &= (P_+)^{\beta_1 \dots} \frac{1}{s} [\frac{1}{2} \gamma_i \gamma_j \delta_{\beta_1}^{\delta_1} \dots - \sum_m \delta_{\beta_1}^{\delta_1} \dots (\delta_{i\beta_m} \delta_j^{\delta_m} - \delta_{j\beta_m} \delta_i^{\delta_m}) \dots] (P_+)^{\gamma_1 \dots} = \frac{1}{s} S_{ij} P_+.
 \end{aligned}$$

Hence the eigenvalues of the magnetic moment are given by

$$(8.10) \quad \langle M_{ij} \rangle = \frac{e}{2m} \frac{\langle S_k \rangle}{s},$$

where $\langle S_k \rangle$ are the eigenvalues of the k -component of the spin, and $i, j, k=1, 2, 3$ cyclically.

This formula has been given by MOLDAUER and CASE⁽⁷⁾, but derived rigorously only for spin $\frac{3}{2}$ (see footnote (19)). PETRAS⁽²¹⁾ gave the correct value for the magnetic moment of spin $\frac{3}{2}$ particles, although the Lagrangian underlying Petras theory is not Hermitian. Recently evidence concerning the spin of the muon has been obtained, which is partly based on Eq. (10). GARWIN, LEDERMAN and WEINRICH⁽²²⁾ have found that the gyromagnetic ratio of the muon is 2.00. The gyromagnetic ratio of particles obeying the Fierz-Pauli equation (1) is given by Eq. (10) to be $1/s$. Hence, if the muon is a Fierz-Pauli particle, its spin must be $\frac{1}{2}$.

9. - Polarization operators.

In addition to the spin projection operators Θ introduced in Sect. 2, projection operators which select certain values of the z -component of the spin are needed. (For convenience polarization is always referred to the z -direction.) These will be referred to as polarization operators. The simplest example is encountered in the case of spin $\frac{1}{2}$. Polarization operators for this case have been used by MICHEL and WIGHTMAN⁽²⁵⁾, and BOUCHIAT and MICHEL⁽²⁶⁾. Defining

$$(9.1) \quad P(s) = \frac{1}{2s} (S_z + s),$$

where S_z is the spin operator with eigenvalues $\pm \frac{1}{2}$, it follows that

$$P(s)\psi(s') = \delta_{ss'}\psi(s'),$$

⁽²⁵⁾ L. MICHEL and A. S. WIGHTMAN: *Phys. Rev.*, **98**, 1190 (1955).

⁽²⁶⁾ C. BOUCHIAT and L. MICHEL: *Compt. Rend. Acad. Sci.*, **243**, 642 (1956). See also C. FRONSDAL and H. ÜBERALL, *Phys. Rev.*, to appear.

where ψ describes a pure spin state. The spin operator $S_{\frac{1}{2}}$ may be taken to be either

$$S'_{\frac{1}{2}} = \frac{1}{4i} S^{\mu\nu} \gamma_{\mu} \gamma_{\nu} \quad \text{or} \quad S''_{\frac{1}{2}} = \frac{i}{2} \gamma_5 S^{\mu} \gamma_{\mu},$$

where, if the momentum is directed along the z -axis

$$S^{\mu\nu} = \delta_1^{\mu} \delta_2^{\nu} - \delta_2^{\mu} \delta_1^{\nu}, \quad S^{\mu} = \left\{ 0, 0, \frac{E}{m}, \frac{p}{im} \right\}.$$

It is easy to see that

$$S'_{\frac{1}{2}}(\mathbf{p} - im) = S''_{\frac{1}{2}}(\mathbf{p} - im)$$

so that the two operators are the same when applied to solutions of the Dirac equation. Previous authors have used the operator $S''_{\frac{1}{2}}$, here the alternative $S'_{\frac{1}{2}}$ will be found advantageous. (The prime will be dropped henceforth.)

The generalization of (1) to the case of a total spin of $n + \frac{1}{2}$ with z -component s is

$$(9.2) \quad P_n(s) = d(n, s) \prod_{s' \neq s} (S_{n+\frac{1}{2}} - s'),$$

where s' runs over the eigenvalues of the z -component of the spin, and

$$(9.3) \quad (S_{n+\frac{1}{2}})_{\alpha_1 \dots \alpha_n}^{\beta_1 \dots \beta_n} = S_{\frac{1}{2}} \delta_{\alpha_1}^{\beta_1} \dots \delta_{\alpha_n}^{\beta_n} + \frac{1}{i} \sum_{m=1}^n \delta_{\alpha_1}^{\beta_1} \dots S_{\alpha_m}^{\beta_m} \dots \delta_{\alpha_n}^{\beta_n}.$$

The normalization factor may be found by noting that

$$S_{n+\frac{1}{2}} P_n(s) = s P_n(s).$$

Hence

$$d(n, s) = \left\{ \prod_{s' \neq s} (s - s') \right\}^{-1},$$

and

$$(9.4) \quad P_n(s) = \left\{ \prod_{s' \neq s} (s - s') \right\}^{-1} \prod_{s' \neq s} (S_{n+\frac{1}{2}} - s').$$

The polarization operator commutes with the spin operator, as is seen by verifying that

$$\begin{aligned} p^{\alpha_1} (S_{n+\frac{1}{2}} \Theta)_{\alpha_1 \dots}^{\beta_1 \dots} &= 0, \\ \gamma^{\alpha_1} (S_{n+\frac{1}{2}} \Theta)_{\alpha_1 \dots}^{\beta_1 \dots} &= 0. \end{aligned}$$

Examples of (4) that will be used in the following section are, for spin $\frac{3}{2}$ and $\frac{5}{2}$, respectively

$$(9.5) \quad P_1(s) = \{2s(s^2 - 9/16s^2)\}^{-1}(S_{\frac{3}{2}}^2 + s)(S_{\frac{3}{2}}^2 - 9/16s^2),$$

$$(9.6) \quad P_2(s) = \{2s(2s^4 - 35s^2/4 + 225/64s^2)\}^{-1}(S_{\frac{5}{2}}^2 + s) \cdot \\ \cdot [S_{\frac{5}{2}}^4 + (s^2 - 35/4)S_{\frac{5}{2}}^2 + 225/64s^2].$$

When only the magnitude, but not the sign of the spin component is relevant, the appropriate projection operators are

$$P_n(|s|) = P_n(s) + P_n(-s) = \left\{ \prod_{|s'| \neq |s|} (s^2 - s'^2) \right\}^{-1} \prod_{|s'| \neq |s|} (S_{n+\frac{1}{2}}^2 - s'^2).$$

10. - Angular distributions.

Calculations on angular distributions of the decay products in interactions involving particles of higher spin will be limited to one particular case: the decay of hyperons through the scheme

$$\mathcal{H} \rightarrow \mathcal{N} + \pi,$$

where \mathcal{H} is a hyperon and \mathcal{N} is a nucleon. Of all the elementary particles which are known or suspected to exist, the hyperons are most likely to have higher spin (^{1,2}).

The angular distributions of the hyperon decay products have been calculated by ADAIR (²⁷) and TREIMAN (²⁸). The present calculations differ in two respects. First, ADAIR and TREIMAN assumed that parity is conserved in the decay. This assumption has been found to be inconsistent with observed facts, in recently performed experiments (³⁰), and will not be made here. Second, earlier calculations have been carried out by means of Clebsch-Gordon coefficients, while the present method employs the spin projection operators and the polarization operators. Because the parity non-conserving interaction has a parity conserving part, a partial comparison with the results of ADAIR and TREIMAN can be made.

(²⁷) R. K. ADAIR: *Phys. Rev.*, **100**, 1540 (1955).

(²⁸) S. B. TREIMAN: *Phys. Rev.*, **101**, 1216 (1956).

(²⁹) C. S. WU, E. AMBLER, R. W. HAYWARD, D. D. HOPPEs and R. P. HUDSON: *Phys. Rev.*, **105**, 1413 (1957); R. L. GARWIN, L. M. LEDERMAN and M. WEINRICH: *Phys. Rev.*, **105**, 1415 (1957).

(³⁰) *Venice Conference*, 1957.

The most general direct interaction is given by the following matrix element (no attention is paid to constant factors, since the interest is in the distributions)

$$K = \bar{\psi}_2(p_2)(1 + a\gamma_5)p_2 \dots p_2 P_n(s)\psi_1(p_1)\varphi(p_1 - p_2),$$

where ψ_1 , ψ_2 and φ are the wave functions of the hyperon (spin $n + \frac{1}{2}$, z -component s), the nucleon and the pion, respectively. Non conservation of parity is caused by interference of the two terms in the factor $(1 + a\gamma_5)$. The transition probability, summed over the spins of the two fermions, is calculated by means of Eq. (2.9), and is found to be given by

$$|K|^2 = \text{Tr}\{p_2 \dots p_2 P_n(s) \Theta(p_1)p_2 \dots p_2 (\mathbf{p}_1 - i\mathbf{m}_1)(1 - a^*\gamma_5)(\mathbf{p}_2 - i\mathbf{m}_2)(1 + a\gamma_5)\}.$$

Using the subsidiary conditions, this can be reduced to

$$(10.1) \quad |K|^2 \approx \left[\frac{1 + aa^*}{2} p_1 \cdot p_2 - \frac{1 - aa^*}{2} m_1 m_2 \right] T_1 + \frac{a + a^*}{2} T_2,$$

where

$$T_1 = \text{Tr}\{p_2 \dots p_2 \prod_{[s' \neq |s|]} (S_{n+\frac{1}{2}}^2 - s'^2) \Theta(p_1)p_2 \dots p_2\},$$

$$T_2 = \text{Tr}\left\{p_2 \dots p_2 \frac{1}{s} S_{n+\frac{1}{2}} \prod_{[s' \neq |s|]} (S_{n+\frac{1}{2}}^2 - s'^2) \Theta(p_1)p_2 \dots p_2 \mathbf{p}_1 \mathbf{p}_2 \gamma_5\right\}.$$

The first trace, T_1 , is the parity conserving part of the angular distribution. The second trace depends on the relative signs of s and the z -component of the nucleon momentum. This correlation between the directions of an axial vector and a polar vector is characteristic of a transition in which parity is not conserved.

The quantities T_1 and T_2 can be calculated by making use of the explicit form for the spin projection operators given by (2.7) and (2.8). The results are as follows: In the rest system of the hyperon Eq. (1) reduces to

$$(10.2) \quad |K|^2 \sim \left[\frac{1 + aa^*}{2} p_1 \cdot p_2 - \frac{1 - aa^*}{2} m_1 m_2 \right] T_1 + \frac{a + a^*}{2} \frac{m_1}{2s} |\vec{p}_2| \cos \vartheta T'_2,$$

where ϑ is the angle between the hyperon spin and the nucleon momentum.

For spin $\frac{1}{2}$, T_1 and T'_2 are equal to unity.

For spin $\frac{3}{2}$

$$(10.3) \quad \begin{cases} T_1 = |\vec{p}_2|^2 \left\{ \frac{7}{6} - \frac{3}{8s^2} - \cos^2 \vartheta \right\}, \\ T'_2 = |\vec{p}_2|^2 \left\{ \frac{19}{6} - \frac{3}{8s^2} - 3 \cos^2 \vartheta \right\}. \end{cases}$$

For spin $\frac{5}{2}$ the results are

$$(10.4) \quad \begin{cases} T_1 = \frac{1}{5} |\vec{p}_2|^4 \left\{ \frac{17}{2} s^2 - \frac{97}{4} + \frac{225}{32s^2} - (8s^2 + 10) \cos^2 \vartheta + 30 \cos^4 \vartheta \right\}, \\ T_2' = \frac{1}{5} |\vec{p}_2|^4 \left\{ \frac{49}{2} s^2 - \frac{17}{4} + \frac{225}{32s^2} - (24s^2 + 150) \cos^2 \vartheta + 150 \cos^4 \vartheta \right\}. \end{cases}$$

If a hyperon is produced in a scalar interaction involving a nucleon and various particles of spin zero, the component of its spin in the direction of flight is $\pm \frac{1}{2}$. If the two components of the hyperon beam—that is the spin $+\frac{1}{2}$ component and the spin $-\frac{1}{2}$ component—decay incoherently and are of equal strength, or if the decay conserves parity (that is, if a is zero), the transition probability is given by T_1 alone. This case was considered by ADAIR⁽²⁷⁾ and TREIMAN⁽²⁸⁾. Setting $s^2 = \frac{1}{4}$ in Eqs. (3) and (4), the result is:

for spin $\frac{3}{2}$

$$T_1 \sim 1 + 3 \cos^2 \vartheta$$

and for spin $\frac{5}{2}$

$$T_1 \sim 1 - 2 \cos^2 \vartheta + 5 \cos^4 \vartheta.$$

These angular distributions are the same as those found by ADAIR and TREIMAN. The more complete results of this work are contained in Eqs. (2), (3) and (4).

* * *

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APPENDIX

The connection between the spinor representation used by FIERZ and PAULI, and the tensor representation employed in the present paper, is illustrated by transforming the spin $\frac{3}{2}$ wave equations from one representation to the other. Even though the spinor representation is the more complicated one, it is useful for gaining new insight into the meaning of the subsidiary conditions, and the nature of the spin projection operator.

The van der Waerden spinors that form a basis for a spin $\frac{3}{2}$ representation of the proper elements of the Lorentz group are of rank 3. There are four kinds: a_{ABC} , $a^{\dot{A}}_{CB}$, $a^{\dot{A}\dot{B}}_{\dot{C}}$ and $a^{\dot{A}\dot{B}\dot{C}}$. The improper elements of the Lorentz group transform a_{ABC} into $a^{\dot{A}\dot{B}\dot{C}}$, $a^{\dot{A}}_{BC}$ into $a^{\dot{A}\dot{B}}_{\dot{C}}$, and vice versa. Thus, to represent the Lorentz group the set a_{ABC} , $a^{\dot{A}\dot{B}\dot{C}}$ or the set $a^{\dot{A}}_{BC}$, $a^{\dot{A}\dot{B}}_{\dot{C}}$ is needed. These representations are reducible. The subsidiary conditions satisfied by the irreducible spin $\frac{3}{2}$ parts are

$$(A-1) \quad a_{ABC}, \quad a^{\dot{A}\dot{B}\dot{C}}, \quad \text{completely symmetric,}$$

$$(A-2) \quad a^{\dot{A}}_{BC} = a^{\dot{A}}_{CB}, \quad a^{\dot{A}\dot{B}}_{\dot{C}} = a^{\dot{B}\dot{A}}_{\dot{C}},$$

$$(A-3) \quad p^B_A a^{\dot{A}}_{BC} = 0, \quad p^C_A a^{\dot{A}\dot{B}}_{\dot{C}} = 0,$$

where

$$p_{A\dot{A}} = p_{\mu} \sigma^{\mu}_{A\dot{A}},$$

and $\sigma_4 = iI$, I being the unit 2-by-2 matrix.

The first set possesses an advantage with regard to simplicity. The *spin projection operator is simply the complete symmetrizer*, and the redundant components may be completely eliminated. This advantage is off-set, however, by the non-existence of a first order wave equation connecting a_{ABC} and $a^{\dot{A}\dot{B}\dot{C}}$. For this reason DIRAC, and all subsequent authors, have considered the second set. The wave equation is

$$(A-4) \quad \begin{cases} m a^{\dot{A}}_{BC} = p_{B\dot{B}} a^{\dot{A}\dot{B}}_{\dot{C}}, \\ m a^{\dot{A}\dot{B}}_{\dot{C}} = p^{\dot{B}B} a^{\dot{A}}_{BC}. \end{cases}$$

DIRAC suggested introducing the electromagnetic interaction by means of the substitution

$$(A-5) \quad p_{\mu} \rightarrow p_{\mu} - ieA_{\mu}.$$

However, as pointed out by FIERZ and PAULI, this leads to serious difficulties. They therefore introduced two auxiliary fields, C_A and $C^{\dot{A}}$, with the help of which the free field equations (2), (3) and (4) can be derived from a single Lagrangian principle. Therealso follows from this that

$$(A-6) \quad C_A = C^{\dot{A}} = 0.$$

The substitution (5) may now be carried out in the Lagrangian, or in the Euler-Lagrange equations. Of Eqs. (2), (3), (4), (6) only Eq. (2) remains valid in the presence of an interaction.

The free field Euler-Lagrange equations are

$$(A-7) \quad \begin{cases} m a^{\dot{A}\dot{B}}_{\dot{C}} = \frac{1}{2}(\delta^{\dot{A}}_{\dot{E}} \delta^{\dot{B}}_{\dot{F}} + \delta^{\dot{B}}_{\dot{E}} \delta^{\dot{A}}_{\dot{F}})(p^{\dot{E}D} a^{\dot{F}}_{CD} + p^{\dot{F}}_C C^{\dot{E}}), \\ m a^{\dot{C}}_{AB} = \frac{1}{2}(\delta^{\dot{C}}_A \delta^{\dot{B}}_B + \delta^{\dot{B}}_B \delta^{\dot{C}}_A)(p_{ED} a^{\dot{C}\dot{B}}_{\dot{F}} + p^{\dot{C}}_F C_E), \\ m C_A = -\frac{1}{2} p_{A\dot{B}} C^{\dot{B}} + \frac{1}{6} p^{\dot{B}}_{\dot{C}} a^{\dot{C}}_{AB}, \\ m C^{\dot{A}} = -\frac{1}{2} p^{\dot{A}B} C_B + \frac{1}{6} p^{\dot{C}}_{\dot{B}} a^{\dot{A}\dot{B}}_{\dot{C}}. \end{cases}$$

Introduce tensor indices by writing

$$a^{\dot{A}\dot{B}}_{\dot{C}} = \sigma^{\mu\dot{A}}_{\dot{C}} a^{\dot{B}}_{\mu}, \quad a^{\dot{A}}_{\mu} = \frac{1}{2} \sigma_{\mu}^{\dot{C}}{}_{\dot{B}} a^{\dot{A}\dot{B}}_{\dot{C}}.$$

Eq. (7) then assumes the form

$$\begin{aligned} m a^{\dot{A}}_{\mu} &= (\eta^{\nu}_{\mu})^{\dot{A}}_{\dot{E}} (p^{\dot{E}\dot{C}} a_{\nu\dot{C}} + p_{\nu} C^{\dot{E}}), \\ m a_{\mu\dot{A}} &= (\eta^{\nu}_{\mu})^{\dot{E}}_{\dot{A}} (p_{\dot{E}\dot{C}} a^{\dot{C}}_{\nu} + p_{\nu} C_{\dot{E}}), \\ m C_{\dot{A}} &= -\frac{1}{2} p_{\dot{A}\dot{B}} C^{\dot{B}} + \frac{1}{3} p^{\mu} a_{\mu\dot{A}}, \\ m C^{\dot{A}} &= -\frac{1}{2} p^{\dot{A}\dot{B}} C_{\dot{B}} + \frac{1}{3} p^{\mu} a^{\dot{A}}_{\mu}, \end{aligned}$$

where

$$\begin{aligned} (\eta^{\nu}_{\mu})^{\dot{A}}_{\dot{E}} &= \frac{1}{2} \sigma_{\mu\dot{B}}^{\dot{C}} \frac{1}{2} (\delta^{\dot{A}}_{\dot{E}} \delta^{\dot{B}}_{\dot{F}} + \delta^{\dot{B}}_{\dot{E}} \delta^{\dot{A}}_{\dot{F}}) \sigma^{\nu\dot{F}}_{\dot{C}} = \frac{1}{2} \delta^{\nu}_{\mu} \delta^{\dot{A}}_{\dot{E}} + \frac{1}{4} \sigma_{\mu\dot{E}}^{\dot{C}} \sigma^{\nu\dot{A}}_{\dot{C}}, \\ (\eta^{\nu}_{\mu})^{\dot{E}}_{\dot{A}} &= \frac{1}{2} \delta^{\nu}_{\mu} \delta^{\dot{E}}_{\dot{A}} + \frac{1}{4} \sigma_{\mu}^{\dot{E}}{}_{\dot{C}} \sigma^{\nu\dot{C}}_{\dot{A}}. \end{aligned}$$

Next introduce the Dirac 4-spinors. Suppressing spinor indices

$$\varphi_{\mu} = \begin{pmatrix} a_{\mu A} \\ a_{\mu}^{\dot{A}} \end{pmatrix}, \quad \psi = \begin{pmatrix} i C_A \\ i C^{\dot{A}} \end{pmatrix},$$

and introducing the γ -matrices

$$\gamma_{\mu} = \frac{1}{i} \begin{pmatrix} & \sigma_{\mu\dot{A}B} \\ \sigma_{\mu}^{\dot{A}B} & \end{pmatrix},$$

the wave equations finally reduce to

$$(A-8) \quad \begin{cases} -im\varphi_{\mu} = (\delta^{\nu}_{\mu} - \frac{1}{4}\gamma_{\mu}\gamma^{\nu})(\mathbf{p}\varphi_{\nu} - p_{\nu}\psi), \\ -im\psi = -\frac{1}{2}\mathbf{p}\psi + \frac{1}{3}p^{\mu}\varphi_{\mu}. \end{cases}$$

Equations (8) may also be written

$$(A-9) \quad (p^{\mu}\Gamma_{\mu} + im)\chi = 0,$$

where

$$\chi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \psi \end{pmatrix}, \quad p^{\mu}\Gamma_{\mu} = \begin{pmatrix} & & & & -p_1 + \frac{1}{4}\gamma_1\mathbf{p} \\ & & & & -p_2 + \frac{1}{4}\gamma_2\mathbf{p} \\ & & & & -p_3 + \frac{1}{4}\gamma_3\mathbf{p} \\ & & & & -p_4 + \frac{1}{4}\gamma_4\mathbf{p} \\ (\delta^{\nu}_{\mu} - \frac{1}{4}\gamma_{\mu}\gamma^{\nu})\mathbf{p} & & & & \\ \hline \frac{1}{3}p^1, \frac{1}{3}p^2, \frac{1}{3}p^3, \frac{1}{3}p^4 & & & & -\frac{1}{2}\mathbf{p} \end{pmatrix}$$

Equation (9) is of the same form as that obtained by GUPTA⁽⁶⁾. Although χ has 20 components, as compared with the 16 components in Gupta's theory, a considerable simplification is achieved in writing the Γ_{μ} matrices in terms of the γ -matrices. In Gupta's theory the four matrices must be given explicitly. It should be emphasized that the three formulations of the Fierz-Pauli theory are completely equivalent. Equation (9) shall not be considered further, since a yet simpler formulation (Eq. (3.18)) exists.

New Geometrical Methods of the Theory of Physical Fields.

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Introduction.

Since the introduction of analytical methods in theoretical physics the application of the geometrical attitude, as well as that of the different methods of elementary and differential geometry in physics, is one of the most important resources of theoretical description. Of course, this does not mean

only the use of geometrical methods to explain explicitly the background of motions of test bodies in space and time, but also the different applications of methods of elementary differential geometry as *e.g.* is well known in the case of kinematics. However, these do not yet mean the «geometrization of physics» according to Riemann's concept elaborated clearly by EINSTEIN in the case of his general theory of relativity.

In the topic of theory of physical fields, as matters stand, there seem to exist two very different ways for the applicability of geometrical methods. One of them refers to the pure description of changes of the field excited in the space-time world in accordance with previous use of the geometrical methods; the other is the direct geometrization of the physical field as elaborated in the case of Einstein's theory of gravitation. To distinguish between these field theories of essentially different kind, it seems to be desirable to use the expressions: *field theory of first and of second kind*, respectively, introduced previously [9]. As mentioned above, in the case of a field theory of second kind the geometrical background is the four-dimensional Euclidian (respectively the pseudo-Euclidian) space and the physical fields are described by potentials (or by other quantities of the field) which are ordinary space-time functions. If on the other hand, the physical properties of the field are, according to the idea of Riemann, characterized by the geometrical structure of the space, the theory of the field is called field theory of first kind. From this point of view Einstein's theory of gravitation is *e.g.* a field theory of first kind; the electromagnetic and mesonic theories, respectively, are field theories of second kind.

Since in the case of the field theories of first kind it has become obvious that the geometrical structure of the four-dimensional Riemannian space is determined by the gravitational field alone, many investigators have carefully been studying the possibility of some unified theory of gravitation and electromagnetism. As these very interesting investigations of the past three decades have shown, practically all generalizations of the four-dimensional Riemannian geometry are suitable to serve as a geometrical basis of such a theory [11, 37]. Consequently numerous theories have been put forward, so that the bare recording itself of the various proposals is a real problem, even without taking theories based on geometries with more than four dimensions into account. In spite of these extensive investigations, a generally accepted final unified theory of fields has not yet been found.

In the course of the investigations in the topic of unified field theory of physical fields mentioned above, quite modern methods of differential geometry have been used which seem to be adequate for other applications, especially, in the case of field theories of second kind too. Therefore, in the following the usual problems of the unified field theories, as well as the geometrization of fields will not be dealt with, but quite another point of view will

be put forward. Namely, some kind of geometrization of a given field theory, or field theories respectively, in the Riemannian sense will not be searched for, but, *some field theories of second kind in general metrical spaces will be elaborated*; i.e. we shall regard the metrical geometrical space in which the field is excited as a generalized one, the geometry of which is determined by an arbitrary metrical ground tensor.

Such investigations were known previously in the case of relativistic electrodynamics in a gravitational field, when the relativistic covariant formalism of the Maxwellian theory was elaborated in the Riemannian space. However, the aim of these investigations is not a pure mathematical generalization of the theories of second kind, but they are supported also from the physical point of view as can be seen from the following.

The metrical point geometry of spaces is the geometrical model for isotropic spaces. If a physical field is excited in such a space, the components of the field depend only on the co-ordinates of the space-time world, i.e. are ordinary point functions. The geometry of the space is pseudo-Euclidian, or Riemannian according to whether the medium which is represented by the geometrical space is homogeneous, or inhomogeneous, respectively. However, *if the medium is anisotropic, the physical properties of the medium are quite different in the various directions, therefore the quantities of the field depend not only on the position co-ordinates of the space, but in every space-time point on the direction co-ordinates too.* This means that we have for every direction another different Riemannian space as geometrical basis for the field theory of second kind; i.e. the suitable geometrical basis for the physical field theory is a special kind of superpositions of Riemannian geometries. This heuristic idea is realized by the geometry of such *line-elements spaces* as the Finslerian geometry [4, 9, 35] or the geometry elaborated previously [10]. Hence, it is obvious that our proposition is of some interest from the point of view of the electromagnetic field theory in anisotropic (and inhomogeneous) dielectrics.

To a certain extent an analogous point of view was put forward previously by W. BLASCHKE [2], as well as by O. VARGA [36] in the case of geometrical optics in anisotropic media.

As a quite different application of the idea put forward above the recognition can be regarded that the general line-element geometry is a natural geometrical basis for the classical bilocal field theory [10].

Now, in this paper we shall first elaborate quite heuristically a new relativistic theory of electromagnetic field in dielectrics, as well as a model example of a classical bilocal theory of fields for the illustration of our general point of view. Then will be given a short review of the general classical theory of physical fields of second kind in general spaces elaborated previously, as well as the differential laws of conservation in such a space.

I. – Relativistic theory of the electromagnetic field in uniformly moving dielectrics.

1. – The physical basis of the new geometrical point of view.

According to Einstein's special theory of relativity it is impossible to fix an absolute frame of reference based on any physical event; that is, all inertial systems are completely equivalent. This means especially in the case of the velocity of light that in all inertial systems this velocity is on the one hand isotropic, on the other hand, a universal constant. However, in the case of propagation of light in dielectrics this is not valid, since the velocity of light in the presence of a dielectricum is already different from that in vacuum: c/n (where c is the velocity of light in vacuo, and n is the index of refraction, respectively); furthermore, a natural frame of reference is distinguished by the presence of dielectricum, namely that in which the dielectricum is at rest. This circumstance is indicated by the special formalism of the theory formulated phenomenologically by H. MINKOWSKI fifty years ago [18, 19].

Minkowski's relativistic theory of the electromagnetic field in uniformly moving dielectrics is still an actual problem of field theory having some open questions such as, *e.g.* the problem connected to the energy-momentum tensor of the field [1, 7, 8, 12, 15-17, 21, 23, 24, 31], as well as the theory in the case of anisotropic dielectrics [13, 14]. The theory which will be put forward in the following has some interest not only from the point of view of the applicability of the theory, but renders possible to throw new light upon the old problem of the theory.

The basic idea of our theory can be formulated as follows: The anisotropy of the propagation of light in anisotropic dielectrics, as well as in moving frame of references relatively to the dielectricum too, is a real physical fact, which cannot be left out of account. If we have, however, the advantage in mind which is assured by the elegant formalism of relativistic electrodynamics in vacuum by the constancy of the velocity of light in all uniformly moving frames of reference, it seems advisable *to endeavour to preserve in the case of dielectrics the constancy of the velocity of light in every direction too, at least in the frame of reference in which the dielectricum is at rest.* So it can namely be hoped that the internal structure of the field will become more evident. This attempt, of course, can only be realized, if *the measuring of length is based on the use of light signals* [29]. This means that instead of the original pseudo-Euclidian metric a so-called *light-metric* is introduced which depends directly on the refractive feature of the dielectricum.

Naturally, at this point the problem immediately arises whether the light-metric of space has a direct physical meaning—that is, it expresses the real geometrical structure of the dielectricum—or is it only a pure technique of the geometrical method? This, however is a problem for philosophy which will not be discussed here in detail. Nevertheless, it may be mentioned that also the first version does not lead to contradictions and seems to be supported by an optical argument dealt with in the following.

It is obvious that this method can also be interpreted—particularly, if the above first point of view is accepted—as an intermediate way between the two formulations of field theory of first and second kind, respectively. The space, the geometrical structure of which is determined by the light-metric, can be regarded as the geometrical model of the dielectricum derived by geometrization of the physical (*i.e.*, that of the refractive) feature of the dielectricum in Riemannian sense. However, this does not mean that this theory is a field theory of first kind, since the physical field itself, which is excited in dielectricum, is not yet geometrized, but is described by different field quantities which are defined in the so-called *model space* introduced above. This model space is from the geometrical point of view in the case of inhomogeneous and anisotropic dielectrics a generalized one, in so far as its metrical ground tensor depends on the point co-ordinates, and in addition on the direction co-ordinates, respectively

A further problem is, of course, whether such a distinguished partial geometrization can be carried out and how far the validity of the theory can be expanded. The difficulties involved thereby are that the index of refraction (respectively the dielectric constant) depends in some cases on the electric field strength. However, if dispersive phenomena are not taken into account and only weak fields are considered as in the case of the validity of classical electrodynamics it is usual—which means that the mentioned constants of the dielectricum can be introduced—our above supposition does not lead to any contradiction. Therefore, in the following the supposition will be put forward that *the index of refraction (respectively the dielectric constant) of the dielectricum does not depend either on the electrical field strength, or on time, but is a function of the space co-ordinates only.*

2. - Geometrical characterization of the model space.

In order to carry out our programme analytically initially it is assumed that in the four-dimensional pseudo-Euclidian space-time world a frame of reference is given in which a dielectricum is at rest having the index of refraction n . The geometrical structure of the space-time world is defined by the metrical ground tensor $\gamma_{\mu\nu}$ the components of which are constant and

e.g. have following numerical values:

$$(2.1) \quad \gamma_{00} = -\gamma_{11} = -\gamma_{22} = -\gamma_{33} = 1, \quad \gamma_{\mu\nu} = 0 \quad (\mu \neq \nu).$$

The co-ordinates of the special frame of reference introduced above are denoted by ξ^μ (*). If the dielectricum is an inhomogeneous one, the velocity of light depends on the space co-ordinates of the dielectricum, *i.e.* $c = c(\xi^1, \xi^2, \xi^3)$. Consequently, the time co-ordinate is $\xi^0 = c(\xi^1, \xi^2, \xi^3) \cdot t$, t being the time of reference.

In the case of inhomogeneous and anisotropic dielectrics, however, the dielectric constant, as well as all quantities of the field, depend on the one hand on the position co-ordinates of space and on the other hand in every space-time point on the directions too.

In order to use the common terms of modern geometry the concepts *line-element space*, as well as *the fundamental element of the line-element space* ought to be introduced.

The fundamental element of the line-element space is a line-element defined by its four-dimensional space co-ordinates x^μ and by a contravariant vector v^μ the direction of which corresponds to that of our line-element. Since only a direction is defined by the vector v^μ , it is evident that the components of v^μ are not independent fulfilling the relation

$$v^\mu v_\mu = \text{const},$$

only their proportion having a meaning. The ensemble of the fundamental elements (x, v) is the so-called line-element space (+).

Nevertheless, let us assume that, as above in the case of homogeneous and isotropic dielectrics, for inhomogeneous and anisotropic dielectrics too the velocity of light depends on the spatial co-ordinates, as well as on the spatial directions of the four-dimensional space-time world of the line-elements, *i.e.* $c = c(\xi^1, \xi^2, \xi^3; v^1, v^2, v^3)$.

Finally, in the case of homogeneous and anisotropic dielectrics (like *e.g.* in crystal optics) the velocity of light depends only on the spatial directions, that is $c = c(v^1, v^2, v^3)$. It is clear, that from the geometrical point of view the adequate space is a special kind of the line-element space introduced above.

Naturally, in the course of the above considerations the dependence of light on the co-ordinates x^μ , as well as on the directions v^μ , respectively, representing the inhomogeneity, and the anisotropy of the dielectrics, can be substituted by the dependence of the dielectric constant and of the magnetic

(*) The convention that greek indices run over the values 0, 1, 2, 3, and the italic ones over the values 1, 2, 3 will be used.

(+) x and v are abbreviations for x^μ and v^μ , respectively.

permeability on the co-ordinates and on the directions too.

For the sake of simplicity and to illustrate clearly the basic idea of the theory to be elaborated it should transiently be supposed that the magnetic permeability of the dielectricum is given by $\mu = 1$ and the dielectric constant by $\varepsilon = \varepsilon(\xi^1, \xi^2, \xi^3)$. Hence, based on the Maxwellian relation $c = c_0/\sqrt{\varepsilon}$, where c_0 is the velocity of light in vacuo.

Hence, the time co-ordinate $\xi^0 = (c_0/\sqrt{\varepsilon})t$ is not independent of the space co-ordinates ξ^1, ξ^2, ξ^3 , it is clear that on the one hand this co-ordinate system is not a natural one and on the other hand the question arises whether the space itself still can remain pseudo-Euclidian. Both difficulties can be solved by introducing a new system of co-ordinates with the transformation of co-ordinates:

$$(2.2) \quad x^0 = \sqrt{\varepsilon(\xi^1, \xi^2, \xi^3)} \cdot \xi^0, \quad x^k = \xi^k, \quad (\varepsilon \geq 1),$$

possessing the inverse transformation

$$\xi^0 = x^0 / \sqrt{\varepsilon(\xi^1, \xi^2, \xi^3)}, \quad \xi^k = x^k.$$

Namely, in this manner an adequate system of co-ordinates is found where $x^0 = c_0 t$, as is familiar in the case of the usual co-ordinate system in vacuum.

It should be mentioned that the transformation of co-ordinates (2.2) is, naturally, not a Lorentzian one, hence the determinant of the transformation does not fulfil the well known condition.

To determine the components of the metrical ground tensor in the new co-ordinate system, that is, to give explicitly the metrical fundamental tensor of the model space mentioned in the previous paragraph, the well known formula of transformation of tensors

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \gamma_{\alpha\beta},$$

has to be used deriving for the covariant components:

$$(2.3) \quad g_{00} = \frac{1}{\varepsilon} \gamma_{00}, \quad g_{0j} = -\frac{1}{\varepsilon} \frac{\partial \ln \sqrt{\varepsilon}}{\partial x^j} x^0 \gamma_{00} + \frac{1}{\sqrt{\varepsilon}} \gamma_{0j},$$

$$g_{jm} = \frac{1}{\varepsilon^2} \frac{\partial \sqrt{\varepsilon}}{\partial x^j} \frac{\partial \sqrt{\varepsilon}}{\partial x^m} (x^0)^2 \gamma_{00} - \frac{1}{\varepsilon} x^0 \left(\gamma_{0m} \frac{\partial \sqrt{\varepsilon}}{\partial x^j} + \gamma_{j0} \frac{\partial \sqrt{\varepsilon}}{\partial x^m} \right) + \gamma_{jm},$$

and for the contravariant components

$$(2.4) \quad g^{00} = \varepsilon \gamma^{00} - \frac{\partial \varepsilon}{\partial \xi^k} \xi^0 \gamma^{0k} + \frac{1}{4\varepsilon} \frac{\partial \varepsilon}{\partial \xi^i} \frac{\partial \varepsilon}{\partial \xi^k} (\xi^0)^2 \gamma^{ik},$$

$$g^{ik} = g \gamma^{ik}, \quad g^{i0} = \sqrt{\varepsilon} \cdot \gamma^{i0} - \frac{1}{2\sqrt{\varepsilon}} \frac{\partial \varepsilon}{\partial \xi^j} (\xi^0) \gamma^{ij},$$

respectively. Finally,

$$(2.5) \quad g \stackrel{\text{def}}{=} \det |g_{\mu\nu}| = -\frac{1}{\varepsilon}.$$

The structure of the model space is perfectly determined by the metrical fundamental tensor. The Riemannian curvature is *e.g.* given by

$$R = \frac{1}{4}\varepsilon^{-6}\{\varepsilon + \frac{1}{4}\varepsilon^{-3}(x^0)^2(\partial_i\varepsilon)(\partial^i\varepsilon)\}[(\partial_r\varepsilon)(\partial^r\varepsilon)]^2 - \frac{1}{2}\varepsilon^{-\frac{3}{2}}x^0(\partial^k\varepsilon)\{R_{0k} + R_{k0}\} - R_r^r,$$

where

$$\partial^\mu \stackrel{\text{def}}{=} g^{\mu\nu} \partial_\nu, \quad \partial_\nu \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\nu},$$

are and $R_{\mu\nu} = R_{\mu\cdot\nu\cdot}^\tau$ is the Einstein-Ricci tensor of the space.

In the course of the above considerations a method of construction for the model space starting from the pseudo-Euclidean space with the metrical fundamental tensor (2.1) was elaborated. This method of construction is, however, a quite general one, in so far as that the formulae (2.3)–(2.5) also preserve their validity for a general metrical fundamental tensor $\gamma_{\mu\nu} = \gamma_{\mu\nu}(x)$, or $\gamma_{\mu\nu} = \gamma_{\mu\nu}(x, v)$, respectively, where again v^μ are the co-ordinates of direction in the space-time point x^μ . In the case of a gravitational field *e.g.* $\gamma_{\mu\nu} = \gamma_{\mu\nu}(x)$ and is determined by the distribution of matter in the space-time world. In the presence of an electromagnetic field without gravitation the metrical fundamental tensor of the original space is determined by the energy-momentum tensor of the field taking Einstein's well known equation of the general theory of relativity into account. In spite of this, as a first approximation, let us suppose that the metrical fundamental tensor of the starting space is defined by (2.1), hence it can be assumed that the influence of the electromagnetic field—being in accordance with our previous supposition a weak one—is negligible (*).

Furthermore, in the following, at the first start, for the sake of additional simplicity it should be supposed that our dielectricum is a homogeneous one having a constant index of refraction and dielectric constant, respectively.

(*) Without going into details it should be mentioned that—as second approximation—it may be assumed that

$$\varepsilon' = \varepsilon/\sqrt{1 + 2\chi/c^2}, \quad \mu' = \mu/\sqrt{1 + 2\chi/c^2},$$

χ being the time-component of the dynamical gravitational potential satisfying the equation

$$g_{00} = 1 + 2\chi/c^2,$$

where g_{00} is determined by Einstein's equations [21].

In this case the equations (2.3) and (2.4) are reduced into

$$(2.6) \quad g_{00} = \frac{1}{\varepsilon}, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{\mu\nu} = 0 \quad (\mu \neq \nu)$$

and

$$(2.7) \quad g^{00} = \varepsilon, \quad g^{11} = g^{22} = g^{33} = -1, \quad g^{\mu\nu} = 0 \quad (\mu \neq \nu)$$

respectively.

It is obvious that the space having the metrical fundamental tensor (2.6) is also a pseudo-Euclidian one, however, the metric is defined by light signals. Hence, the velocity of light in a dielectricum with a constant dielectric constant is $c = c_0/\sqrt{\varepsilon}$, the metrical fundamental form is given by

$$(2.8) \quad ds^2 = F^2(x, dx) \stackrel{\text{def}}{=} g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{\varepsilon} (x^0)^2 - [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] = \\ = \frac{c_0^2}{\varepsilon} dt^2 - [(dx^1)^2 + (dx^2)^2 + (dx^3)^2],$$

as was expected.

The transformation (2.2) of the co-ordinates is in this case a pure proportional transformation of the gauge along the axis of time.

Now, it should be mentioned that the previous considerations are valid, so far, in the special co-ordinate system K in which the dielectricum is at rest. To calculate the metrical fundamental tensor in a new frame of reference K' moving uniformly with the velocity $w = \{w_1, w_2, w_3\}$ relative to K the proper Lorentz-transformation

$$(2.9) \quad dx^{\mu'} = a_{\mu}^{\mu'} dx^\mu; \quad a_{\mu}^{\mu'} a_{\nu}^{\mu} = \delta_{\nu}^{\mu'}, \quad \det |a_{\nu}^{\mu'}| = +1$$

having the matrix of transformation

$$a_{\nu}^{\mu'} = \begin{pmatrix} \kappa & -\kappa w_1/c & -\kappa w_2/c & -\kappa w_3/c \\ -\kappa w_1/c & 1 + (\kappa - 1)w_1^2/w^2 & (\kappa - 1)w_1 w_2/w^2 & (\kappa - 1)w_1 w_3/w^2 \\ -\kappa w_2/c & (\kappa - 1)w_2 w_1/w^2 & 1 + (\kappa - 1)w_2^2/w^2 & (\kappa - 1)w_2 w_3/w^2 \\ -\kappa w_3/c & (\kappa - 1)w_3 w_1/w^2 & (\kappa - 1)w_3 w_2/w^2 & 1 + (\kappa - 1)w_3^2/w^2 \end{pmatrix} \\ \kappa = \{1 - w^2/c^2\}^{-\frac{1}{2}}$$

should be used; finally, for the contravariant components of the metrical fundamental tensor of the model space

$$g^{00'} = \frac{\varepsilon - w^2/c^2}{1 - w^2/c^2}, \quad g^{0i'} = \frac{1 - \varepsilon}{1 - w^2/c^2} \frac{w_i}{c}, \quad g^{ik'} = -\delta_{ik} + \left(1 - \frac{1 - \varepsilon w^2/c^2}{1 - w^2/c^2}\right) \frac{w_i w_k}{w^2},$$

is obtained.

It is obvious, of course, that the formulae (2.3)–(2.9) at the limit $\varepsilon \rightarrow 1$ —i.e. in vacuo—become reduced to the well known ones.

The above method of construction renders possible, in principle, the explicit determination of the metrical ground tensor of the model space, if the dependence of the dielectric constant on the spatial co-ordinates is known. Occasionally, the practical carrying out of this procedure is quite simple, on other occasions it is, however, complicated, inasmuch as the explicit dependence of the dielectric constant in cases of inhomogeneous and anisotropic dielectrics is not known. This problem will be discussed in the next paragraph too, nevertheless, now another method directly applicable, at least, whenever the dielectricum is homogeneous and isotropic, will be given which is, however, also from the theoretical point of view of some interest.

By means of a covariant vector v^μ , introduced above for all directions of the space, a *scalar fundamental function*

$$(2.10) \quad \mathfrak{F}(x, v) \stackrel{\text{def}}{=} \left\{ \frac{1}{\varepsilon} (v^0)^2 - \delta_{ik} v^i v^k \right\}^{\frac{1}{2}},$$

can be defined. It is easy to see that on the one hand, the measuring of the length can be characterized by \mathfrak{F} , hence

$$ds = \mathfrak{F}(x, dx)$$

and on the other hand, the fundamental function \mathfrak{F} has the following property

$$(2.11) \quad g_{\mu\nu} = \frac{1}{2} \frac{\partial^2 \mathfrak{F}^2}{\partial v^\mu \partial v^\nu},$$

i.e. the metrical fundamental tensor is determined by the introduced fundamental function. However, it should repeatedly be made perfectly clear that this last relation is only valid in the case of dielectrics having a dielectric constant independent from v .

The relation (2.11) is characteristic of the Finslerian geometry [4, 9, 35], consequently the geometry of our model space is a special case of the Finslerian one.

Often, also in the case of an inhomogeneous, as well as an anisotropic dielectricum it is theoretically possible to introduce such a fundamental function. Generally, however, an adequate geometrical basis for the field theory is, when such a fundamental function cannot be introduced, the *general line-element geometry* [10] being an immediate generalization of the Finslerian one.

3. - Indicatrix of space and Fresnel's ellipsoid of the dielectricum.

To characterize the structure of the model space at a point $P(x)$, let us consider all directions crossing over P . The end point of the unit vectors directed towards the different directions starting from P form a hyper-surface in the four-dimensional model space having the equation

$$(3.1) \quad \mathfrak{F}(x, \xi) = 1,$$

where

$$\xi^\mu = v^\mu / \mathfrak{F}$$

is the vector of unit length in the direction v^μ . This surface is the so-called *Carathéodorian indicatrix of the space* at the point P .

Let the indicatrix (3.1) be cut by the hyper-plane

$$(3.2) \quad \xi^0 = \sqrt{1 + \varepsilon},$$

then the equation of the surface of intersection is given by

$$(3.3) \quad \delta_{ik} \xi^i \xi^k = \frac{1}{\varepsilon},$$

being just the *Fresnel ellipsoid of the dielectricum*.

This is, otherwise, obvious from the method of construction and a direct consequence of the introduction of the light-metric in our model space.

The correspondence between the indicatrix of the space and the Fresnel ellipsoid of the dielectricum in the case discussed above is quite trivial, but in the general case for the construction of the metrical fundamental tensor of the model space it is of good service to us. Namely, *e.g.* in the case of a homogeneous, but anisotropic dielectricum the problem to determine the indicatrix in such a way that its cutting with a suitable hyper-plane should be

$$(3.4) \quad \varepsilon_{ik}^* \xi^i \xi^k = 1$$

may be put forward, where ε_{ik}^* is the known dielectric tensor of the dielectricum.

In order to carry out this programme we should start from the assumption that the anisotropy of the dielectricum is characterized by the functional dependence of the dielectric constant on the spatial direction co-ordinates, *i.e.* $\varepsilon = \varepsilon(v)$. According to the method of construction elaborated in the previous paragraph for the components of the metrical fundamental tensor of the

model space the implicit formulae

$$g_{00} = \frac{1}{\varepsilon(v)}, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{\mu\nu} = 0 \quad (\mu \neq \nu),$$

can be obtained. With the help of this metrical fundamental tensor the fundamental function

$$\mathfrak{F}(v) = \left\{ \frac{1}{\varepsilon(v)} (v^0)^2 - \delta_{ik} v^i v^k \right\}^{\frac{1}{2}},$$

may be introduced and the corresponding Carathéodorian indicatrix of the space is given by

$$\frac{1}{\varepsilon(v)} (\xi^0)^2 - \delta_{ik} \xi^i \xi^k = 1.$$

By cutting this indicatrix by the hyper-plane

$$\xi^0 = \sqrt{1 + \varepsilon(v)},$$

the equation of the surface of intersection

$$(3.5) \quad \delta_{ik} \xi^i \xi^k = \frac{1}{\varepsilon(v)},$$

is obtained. On the basis of the construction elaborated above in the isotropic case this surface must be the Fresnel ellipsoid of the dielectricum. Therefore, the equations (3.5) and (3.4) may be set side by side. In order to render possible the determination of the explicit dependence of ε on v , the ellipsoid (3.4) will be cut by the straight line

$$\xi^k = v^k \tau,$$

where the direction of the straight line is determined by v^k and τ is a parameter. Hence, by substitution

$$(\varepsilon_{ik}^* \xi^i \xi^k) \tau^2 = 1$$

is obtained, and the co-ordinates of the point of the ellipsoid (3.4) are given by

$$\xi^k = v^k / \{\varepsilon_{rs}^* v^r v^s\}^{\frac{1}{2}}.$$

Nevertheless, the same Fresnel ellipsoid is represented by the equations (3.4) and (3.5), therefore,

$$\frac{1}{\varepsilon(v)} = \frac{\delta_{ik} v^i v^k}{\varepsilon_{rs}^* v^r v^s},$$

is obtained. Consequently the fundamental function of the model space is given explicitly by

$$\mathfrak{F}(v) = \left\{ \frac{\delta_{ik} v^i v^k}{\varepsilon_{rs}^* v^r v^s} (v^0)^2 - \delta_{pq} v^p v^q \right\}^{\frac{1}{2}}.$$

In geometry usually such a space is called Minkowskian being a special case of the Finslerian space [34].

The metrical fundamental tensor of this Minkowskian space is determined by (2.11) and is given by

$$g_{00} = \frac{\delta_{ik} v^i v^k}{\varepsilon_{rs}^* v^r v^s}, \quad g_{0j} = \frac{\partial g_{00}}{\partial v^j} v^0, \quad g_{ij} = \frac{1}{2} \frac{\partial^2 g_{00}}{\partial v^i \partial v^j} (v^0)^2 - \delta_{ij}.$$

In this manner, the geometrization of the dielectric (respectively the refraccional) property of the dielectricum is carried out and we can proceed to the elaboration of the theory of the electromagnetic field in the model space introduced above. The theory, however, will only be discussed, in the homogeneous and isotropical case, the formulation of which is quite simple, well perspicuous and very suitable for direct comparison with the previous theories.

4. - Theory of the electromagnetic field in dielectrica with dielectric constant $\varepsilon = \text{const.}$ and magnetic permeability $\mu = 1$.

In a dielectricum with the dielectric constant $\varepsilon = \text{const.}$ and with the magnetic permeability $\mu = 1$, *i.e.* in the case of a homogeneous and isotropic dielectricum with vanishing magnetic susceptibility, the metrical fundamental tensor of the model space is obtained by (2.6).

The quantities of the electromagnetic field are derived from the four-potential $\Phi_\mu = \{-\varphi, \mathfrak{A}\}$ fulfilling the D'Alembert equations

$$(4.1) \quad \square \Phi_\mu = -s_\mu,$$

where

$$\square \stackrel{\text{def}}{=} g^{\mu\nu} \partial_\mu \partial_\nu \equiv \frac{\varepsilon}{c_0^2} \partial_t^2 - \{\partial_1^2 + \partial_2^2 + \partial_3^2\} \equiv \frac{\varepsilon}{c_0^2} \partial_t^2 - \Delta,$$

and $s^\mu = \{\varrho, i/c_0\}$, denoting φ and \mathfrak{A} the scalar and vector potential respectively; furthermore, ϱ and i being the macroscopic charge and current densities.

The potential Φ_μ , as usually, has to fulfil the Lorentz condition

$$(4.2) \quad \partial_\mu \Phi^\mu = 0.$$

Attention should be paid to the fact that the contravariant component of the current-density vector is defined directly by φ and i —as it is usual—and, therefore, its covariant component occurring in the equation (4.1) is obtained by

$$(4.3) \quad s_\mu = g_{\mu\varrho} s^\varrho = \{\varrho/\varepsilon, -i/c_0\}.$$

In the classical theory of Minkowski the electromagnetic field in dielectrics is characterized by two antisymmetrical tensors $F_{\mu\nu}$ and $G_{\mu\nu}$, respectively, having the components

$$(4.4) \quad \mathfrak{E} \equiv \{F_{10}, F_{20}, F_{30}\}, \quad \mathfrak{B} \equiv \{F_{23}, F_{31}, F_{12}\} = \mathfrak{S}$$

and

$$\mathfrak{D} \equiv \{G_{10}, G_{20}, G_{30}\}, \quad \mathfrak{H} \equiv \{G_{23}, G_{31}, G_{12}\},$$

where \mathfrak{E} , \mathfrak{D} , \mathfrak{H} and \mathfrak{B} denote the electric field strength, electric displacement, magnetic field strength and magnetic induction, respectively.

In the theory elaborated here, however, the electromagnetic field is described by a single field tensor derived from the four-potential by

$$(4.5) \quad F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu$$

having the covariant components (4.4). But its contravariant component is defined by

$$(4.6) \quad F^{\mu\nu} = g^{\varrho\nu} g^{\sigma\mu} F_{\varrho\sigma},$$

that is, owing to (2.6)

$$(4.7) \quad \mathfrak{D} \equiv \{F^{01}, F^{02}, F^{03}\}, \quad \mathfrak{H} = \{F^{23}, F^{31}, F^{12}\},$$

are obtained.

Already at this point the advantage of the new geometrical method may be seen. The intrinsic structure of the electromagnetic field is expressed far more elegantly and naturally, furthermore, the formulation of the theory becomes more simple.

Owing to definition (4.5) for the field tensor $F_{\mu\nu}$ the identity

$$(4.8) \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

is obtained corresponding to the second group of Maxwell's equations.

In order to carry out the deductive formulation of the theory in the case of empty space (*i.e.* $\varrho = 0$, $i = 0$) the Lagrangian

$$(4.9) \quad L_0 \stackrel{\text{def}}{=} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \equiv \frac{1}{2} (\mathfrak{B}\mathfrak{H} - \mathfrak{E}\mathfrak{D})$$

is introduced being formally, however, only formally, equivalent with the Lagrangian of the field in vacuo (*). This means that if the model space is introduced the field theory in dielectrics represents a direct generalization of the theory in vacuo.

If s^μ does not vanish, the Lagrangian of the field is given by

$$(4.10) \quad L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \Phi_\mu s^\mu,$$

where

$$\Phi_\mu s^\mu = (\varrho/\varepsilon)\varphi + \frac{1}{c_0} (i\mathfrak{H})$$

represents the energy of interaction between field and macroscopic charge and current densities, respectively. However, the charge density occurring in the Lagrangian is the density of the so-called free-charge ϱ/ε , as was expected.

As is well known the field equation can be deduced by variation of the integral of action

$$(4.11) \quad I = \int_{\Omega} \sqrt{|g|} L d^4x,$$

varying the potentials Φ_μ for the fixed region Ω of integration, subject to the restriction that the variation of Φ_μ and its first derivatives at the boundary of the domain of integration vanish.

The integral of action given by the definition (4.11) represents the general case. If the space is, however, the special one with the metrical ground tensor (2.6) $\varepsilon = \text{const}$, the volume integral of L is again an integral invariant.

By carrying out the variation in the usual way the field equation

$$(4.12) \quad \partial_\nu F^{\mu\nu} = s^\mu$$

is obtained, representing the first group of Maxwell's equations.

Based on (4.12)

$$(4.13) \quad \partial_\mu s^\mu = 0$$

is obtained representing the so-called equation of continuity, that is the differential law of conservation of electric charge.

Now, let us interrupt for a few moments the elaboration of the theory and return again to the problem of determination of the metrical fundamental tensor

(*) Naturally, for the limit $\varepsilon \rightarrow 1$ the whole formulation of the theory is exactly equivalent with that in vacuum.

of the model space. Namely, on the basis of equation (4.6) the mentioned problem can be solved very strikingly. This new method has the advantage that it makes obvious that equation (4.6) is essentially a new covariant formulation of the constitutive equations determining the well known relations between the electric field strength and the electric displacement vector, furthermore between the magnetic field strength and the magnetic induction vector, respectively, in four-dimensional language.

Let u^α be the four-vector of the velocity of the dielectricum having thus in the rest system of the dielectricum the components $\{c_0, 0, 0, 0\}$. Then, in the Minkowskian theory the classical constitutive equations of Maxwell's theory

$$\mathfrak{D} = \varepsilon \mathfrak{E}, \quad \mathfrak{H} = \frac{1}{\mu} \mathfrak{B},$$

can be given in the form

$$G_{\alpha\beta} u^\beta = \varepsilon F_{\alpha\beta} u^\beta$$

$$G_{\beta\lambda} u_\alpha + G_{\lambda\alpha} u_\beta + G_{\alpha\beta} u_\lambda = \frac{1}{\mu} \{ F_{\beta\lambda} u_\alpha + F_{\lambda\alpha} u_\beta + F_{\alpha\beta} u_\lambda \}.$$

By contraction of the second equation by u^λ

$$G_{\beta\lambda} u^\lambda u_\alpha + G_{\lambda\alpha} u^\lambda u_\beta + c_0^2 G_{\alpha\beta} = \frac{1}{\mu} \{ F_{\beta\lambda} u^\lambda u_\alpha + F_{\lambda\alpha} u^\lambda u_\beta + c_0^2 F_{\alpha\beta} \},$$

and finally

$$G^{\alpha\beta} = \frac{1}{\mu} \left\{ \gamma^{\alpha\varrho} \gamma^{\beta\sigma} + \frac{\varepsilon\mu - 1}{c_0^2} [u^\alpha \gamma^{\beta\sigma} u^\varrho - u^\beta \gamma^{\alpha\sigma} u^\varrho] \right\} F_{\varrho\sigma} =$$

$$= \frac{1}{\mu} \left\{ \gamma^{\alpha\varrho} \gamma^{\beta\sigma} + \frac{\varepsilon\mu - 1}{c_0^2} [u^\alpha \gamma^{\beta\sigma} u^\varrho + u^\beta \gamma^{\alpha\sigma} u^\varrho] \right\} F_{\varrho\sigma},$$

is obtained. Taking into account that $F_{\varrho\sigma}$ is antisymmetrical in its indices, the coefficient of $F_{\varrho\sigma}$ is reducible and

$$(4.14) \quad G^{\alpha\beta} = \frac{1}{\mu} \left(\gamma^{\alpha\varrho} + \frac{\varepsilon\mu - 1}{c_0^2} u^\alpha u^\varrho \right) \left(\gamma^{\beta\sigma} + \frac{\varepsilon\mu - 1}{c_0^2} u^\beta u^\sigma \right) F_{\varrho\sigma},$$

is obtained. Now, let the equations (4.6) and (4.14) be set side by side, then it becomes obvious that between the metrical fundamental tensor $g^{\alpha\beta}$ of the model space and the metrical fundamental tensor $\gamma^{\alpha\beta}$ of the original pseudo-Euclidian space and the constants of the dielectricum ε and μ , respectively, the relation exists:

$$(4.15) \quad g^{\alpha\beta} = \frac{1}{\sqrt{\mu}} \left\{ \gamma^{\alpha\beta} + \frac{\varepsilon\mu - 1}{c_0^2} u^\alpha u^\beta \right\}.$$

Apparently, whenever $\mu = 1$, $g^{\alpha\beta}$ is equivalent with the metrical fundamental tensor introduced above.

On the basis of the above considerations it seems possible that also in the case of homogeneous and anisotropic dielectrics the same method is suitable for the determination of the metrical fundamental tensor of the model space. However, this is unfortunately not the case, or, at least, the investigations in this direction have so far not produced the expected result. Nevertheless, the method will be discussed in this case too, since it, as well as the detail-results are also interesting in themselves.

The problem will be discussed in the case when from the electric point of view the dielectricum is anisotropic having the dielectric tensor ε_{ik}^* , but, from the magnetic point of view homogeneous, that is $\mu = \text{const}$. Then, the constitutive equations of Maxwell's theory

$$D_i = \sum_{k=1}^3 \varepsilon_{ik}^* E_k, \quad \mathfrak{D} = \frac{1}{\mu} \mathfrak{B},$$

can be given in four-dimensional language in the following form

$$(4.16) \quad (G^{\alpha\beta} \gamma_{\beta\tau} + \varepsilon^{\alpha\varrho} F_{\varrho\tau}) u^\tau = 0.$$

and

$$(4.17) \quad G^{\beta\lambda} u^\alpha + G^{\lambda\alpha} u^\beta + G^{\alpha\beta} u^\lambda = \frac{1}{\mu} \{ \gamma^{\beta\varrho} \gamma^{\lambda\sigma} u^\alpha + \gamma^{\lambda\varrho} \gamma^{\alpha\sigma} u^\beta + \gamma^{\alpha\varrho} \gamma^{\beta\sigma} u^\lambda \} F_{\varrho\sigma},$$

respectively, u^τ being the above introduced four-velocity of the dielectricum and

$$\varepsilon^{00} = 1, \quad \varepsilon^{0k} = 0, \quad \varepsilon^{ik} = \varepsilon_{ik}^*$$

are supposed. By contraction with $\gamma_{\lambda\tau} u^\tau$ taking (4.17) and (4.16) into account

$$\varepsilon^{\beta\varrho} F_{\varrho\tau} u^\tau u^\alpha + \varepsilon^{\alpha\varrho} u^\tau u^\beta + c_0^2 G^{\alpha\beta} = \frac{1}{\mu} \{ \gamma^{\beta\varrho} u^\sigma u^\alpha + \gamma^{\alpha\sigma} u^\varrho u^\beta + c_0^2 \gamma^{\alpha\varrho} \gamma^{\beta\sigma} \} F_{\varrho\sigma},$$

is obtained, therefore

$$(4.18) \quad G^{\alpha\beta} = \frac{1}{\mu} \left\{ \gamma^{\alpha\varrho} \gamma^{\beta\sigma} + \frac{1}{c_0^2} [(\mu \varepsilon^{\alpha\varrho} - \gamma^{\alpha\varrho}) u^\beta - (\mu \varepsilon^{\beta\varrho} - \gamma^{\beta\varrho}) u^\alpha] u^\sigma \right\} F_{\varrho\sigma},$$

is the expected relation between the two antisymmetrical tensors of the field. Now, the equations (4.18) and (4.14) can be set side by side, however, the coefficients of $F_{\varrho\sigma}$ seem to be irreducible against such a factorization carried out in the case of (4.14). Therefore in this way the introduction of a metrical fundamental tensor is hindered.

So, in this case for the construction of the metrical fundamental tensor of the model space only the method elaborated in the previous paragraph is possible. However, the close analogy with the classical theory is lost.

5. - The electromagnetic field in an arbitrary homogeneous and isotropic dielectricum.

Now, the considerations of the previous paragraph should be generalized and the meaning of the deduced formulae elaborated in three-dimensional language too.

In the general case of isotropic dielectrics and paramagnetic dielectrics, that is, for $\varepsilon = \text{const.}$ and $\mu = \text{const.}$, respectively, the contravariant component of the metrical fundamental tensor of the model space is given by (4.15):

$$g^{00} = \varepsilon\sqrt{\mu}, \quad g^{11} = g^{22} = g^{33} = -1/\sqrt{\mu}, \quad g^{\mu\nu} = 0 \quad (\mu \neq \nu).$$

It can be easily verified, that for the covariant component of the metrical fundamental tensor

$$(5.1) \quad g_{00} = 1/\varepsilon\sqrt{\mu}, \quad g_{11} = g_{22} = g_{33} = -\sqrt{\mu}, \quad g_{\mu\nu} = 0 \quad (\mu \neq \nu)$$

and for the determinant

$$g = \det|g| = -\frac{\mu}{\varepsilon},$$

is obtained.

By carrying out the Lorentz transformation (2.9), the components of the metrical fundamental tensor of the model space in the frame of reference moving with the velocity $w = \{w_1, w_2, w_3\}$ uniformly relative to the dielectricum are given by

$$\begin{aligned} \bar{g}^{00} &= \frac{1 - (1/\varepsilon\mu)(w^2/c_0^2)}{1 - w^2/c_0^2} \varepsilon\sqrt{\mu}; & \bar{g}^{0i} &= \frac{1 - 1/\varepsilon\mu}{1 - w^2/c_0^2} \varepsilon\sqrt{\mu} \frac{w_i}{c_0}; \\ \bar{g}^{ik} &= -\varepsilon\sqrt{\mu} \left\{ \frac{\delta_{ik}}{\varepsilon\mu} - \frac{w_i w_k}{w^2} \left[\frac{1}{\varepsilon\mu} - \frac{1/\varepsilon\mu - w^2/c_0^2}{1 - w^2/c_0^2} \right] \right\}. \end{aligned}$$

In this manner the electric and magnetic anisotropy of the model space originating from the velocity of the frame of reference is determined.

The covariant component of the field tensor is again defined by (4.5) and (4.4), respectively, and for its contravariant component owing to (4.6)

$$\mathfrak{D} = \{F^{01}, F^{02}, F^{03}\}, \quad \mathfrak{S} = \{F^{23}, F^{31}, F^{12}\}$$

is obtained. Equation (4.6) is equivalent with the constitutive equation (4.14) of the dielectricum.

The Lagrangian of the field in empty space is given by

$$(5.2) \quad L_0 = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \equiv \frac{1}{2} \{ \mathfrak{B} \mathfrak{E} - \mathfrak{C} \mathfrak{D} \}$$

and, if the four-current density does not vanish, by

$$(5.3) \quad L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \Phi_\mu s^\mu,$$

respectively, which are only formally equivalent with (4.9) and (4.10).

The D'Alembert equation of the potentials has the form

$$(5.4) \quad \square \Phi_\alpha \equiv g^{00} \frac{1}{c_0^2} \partial_t^2 \Phi_\alpha + g^{ik} \partial_i \partial_k \Phi_\alpha \equiv \frac{\varepsilon \sqrt{\mu}}{c_0^2} \partial_t^2 \Phi_\alpha - 1/\sqrt{\mu} \Delta \Phi_\alpha = -s_\alpha.$$

But

$$s_\alpha = g_{\alpha\varrho} s^\varrho \equiv \{\varrho/\varepsilon \sqrt{\mu}, -i\sqrt{\mu}/c_0\},$$

therefore, in three-dimensional language the components of Φ_α must fulfil the equations

$$\Delta \varphi - \frac{\varepsilon \mu}{c_0^2} \partial_t^2 \varphi = -\varrho/\varepsilon, \quad \Delta \mathfrak{A} - \frac{\varepsilon \mu}{c_0^2} \partial_t^2 \mathfrak{A} = -\mu i/c_0.$$

Similarly, field equation (4.12) represents the Maxwell equations

$$\operatorname{div} \mathfrak{D} = \varrho, \quad \operatorname{curl} \mathfrak{E} - \frac{1}{c_0} \partial_t \mathfrak{D} = i/c_0,$$

as was expected.

The D'Alembert equations for the potentials are compatible with the field equations (4.12) then and only then, if the potentials Φ_α fulfil the Lorentz condition (4.2):

$$\partial_\alpha \Phi^\alpha = g^{\alpha\beta} \partial_\alpha \Phi_\beta = \frac{\varepsilon \sqrt{\mu}}{c_0} \partial_t \Phi_0 - \frac{1}{\sqrt{\mu}} \{ \partial_1 \Phi_1 + \partial_2 \Phi_2 + \partial_3 \Phi_3 \} = 0,$$

being adequate in three-dimensional language with

$$\operatorname{div} \mathfrak{A} + \frac{\varepsilon \mu}{c_0} \partial_t \varphi = 0.$$

The equation of continuity (4.13) preserves its formal validity too, however, in three-dimensional language it has the meaning

$$\partial_t \varrho + \operatorname{div} \mathfrak{i} = 0.$$

The results of the two last paragraphs can be summarized as follows: *the formalism of the electrodynamic field in vacuo preserves its formal validity in homogeneous and isotropic dielectrics too, if the new geometrical aspect is introduced.* Naturally, it is true that at first sight, even if the complete formalism of the theory is more simple than the previous Minkowskian one, these results seem only to be formal, however, as will be shown in the next paragraph, it also has its advantages from the physical point of view.

6. — Differential law of conservation. Energy-momentum tensor of the field. Four-force density.

As is well known, based on the infinitesimal transformation of co-ordinates

$$(6.1) \quad x^{\mu'} = x^{\mu} + \eta \xi^{\mu}(x)$$

(η being an infinitesimal parameter and $\xi^{\mu}(x)$ an arbitrary covariant vector which is a continuous and limited function of the co-ordinates x), some identities can be deduced which from the physical point of view can be interpreted as the differential laws of conservation of the field [25].

This method is very familiar in the topic of field theory and has previously been discussed in detail [7, 8], furthermore it will be dealt with too in the course of the elaboration of the general theory in Chapter IV; therefore, it will be elaborated here only in broad outlines.

If the variation of the Lagrangian density brought about by the change of co-ordinates is investigated the explicit dependence of the Lagrangian on the covariant components of the metrical ground tensor $g^{\alpha\beta}$, on the four-potential Φ_{α} and its derivatives $\Phi_{\alpha,\beta}$ and on the current-density $\sqrt{|g|}s^{\alpha}$, respectively, must be taken into account. Owing to (4.10) it may be presumed that \mathfrak{L} has the explicit form:

$$(6.2) \quad \mathfrak{L} = \sqrt{|g|} \left\{ \frac{1}{2} g^{\alpha\sigma} g^{\beta\sigma} F_{\alpha\beta} F_{\sigma\sigma} - \Phi_{\alpha} s^{\alpha} \right\} \equiv \sqrt{|g|} L(g^{\alpha\beta}; \Phi_{\alpha}; \Phi_{\alpha,\beta}; s^{\alpha}).$$

Assuming that the variation of $g^{\alpha\beta}$, Φ_{α} and s^{α} , respectively, is carried out by arbitrary, as well as independent ways, the *total variation of I* —introduced in (5.3)—subjected to the restrictions that the ξ^{λ} -s of the infinitesimal transformation (6.1) at the boundary of the domain of integration Ω vanish, is given by

$$\delta I = \int_{\Omega} \delta^* \mathfrak{L} d^4x,$$

where $\delta^* \mathfrak{L}$ is the so-called *local variation of \mathfrak{L}* :

$$\delta^* \mathfrak{L} \stackrel{\text{def}}{=} \mathfrak{L}'(x) - \mathfrak{L}(x),$$

that is,

$$\delta\Omega^* = L\delta^*(\sqrt{|g|}) + \sqrt{|g|} \left\{ \frac{\partial L}{\partial g^{\alpha\beta}} \delta^* g^{\alpha\beta} + \frac{\partial L}{\partial \Phi_\alpha} \delta^* \Phi_\alpha + \frac{\partial L}{\partial \Phi_{\alpha,\tau}} \delta^* \Phi_{\alpha,\tau} + \frac{\partial L}{\partial s^\alpha} \delta^* s^\alpha \right\},$$

and, finally, by partial integration

$$\delta I = \int_{\Omega} \sqrt{|g|} \left\{ \frac{1}{2} \left[2 \frac{\partial L}{\partial g^{\alpha\beta}} - g_{\alpha\beta} L \right] \delta^* g^{\alpha\beta} + \left[\frac{\partial L}{\partial \Phi_\alpha} - \partial_\tau \frac{\partial L}{\partial \Phi_{\alpha,\tau}} \right] \delta^* \Phi_\alpha + \frac{\partial L}{\partial s^\alpha} \delta^* s^\alpha \right\} d^4x,$$

is obtained. Based on the explicit form of the Lagrangian (6.2), we have

$$\begin{aligned} \delta I = \int_{\Omega} \sqrt{|g|} \left\{ \left[\frac{1}{2} (F_{\alpha\varrho} F_{\beta}^{\varrho} - \frac{1}{4} g_{\alpha\beta} F_{\varrho\sigma} F^{\varrho\sigma}) + \frac{1}{2} g_{\alpha\beta} \Phi_{\tau} s^{\tau} \right] \delta^* g^{\alpha\beta} - \right. \\ \left. - [\partial_\tau F^{\alpha\tau} - s^\alpha] \delta^* \Phi_\alpha - \Phi_\alpha \delta^* s^\alpha \right\} d^4x. \end{aligned}$$

Introducing the tensor

$$(6.3) \quad T_{\alpha\beta} \stackrel{\text{def}}{=} - \left\{ F_{\alpha\varrho} F_{\beta}^{\varrho} - \frac{1}{4} g_{\alpha\beta} F_{\varrho\sigma} F^{\varrho\sigma} \right\}$$

and considering the field equation (4.12), as well as the relations (Sect. 13):

$$\begin{aligned} \delta^* g^{\alpha\beta} &= -\eta \{ (\partial_\lambda g^{\alpha\beta}) \xi^\lambda - (\partial_\lambda \xi^\alpha) g^{\lambda\beta} - (\partial_\lambda \xi^\beta) g^{\alpha\lambda} \} + O(\eta^2), \\ \delta^* s^\alpha &= -\eta \{ (\partial_\lambda s^\alpha) \xi^\lambda - (\partial_\lambda \xi^\alpha) s^\lambda \} + O(\eta^2), \end{aligned}$$

by partial integration and on the basis of the definition (4.5) of $F_{\lambda\tau}$

$$\delta I = \eta \int_{\Omega} \sqrt{|g|} \left\{ \partial^\lambda T_{\tau}^{\lambda} + F_{\lambda\tau} s^{\tau} \right\} \xi^{\tau} d^4x,$$

is obtained. However, I is an invariant of the arbitrary changes of co-ordinates, therefore, it is stationary:

$$\delta I \equiv 0.$$

This is an identity for arbitrary ξ^α -s and for arbitrary choice of the integration region, thus we have finally

$$(6.4) \quad -\partial_\lambda T_{\tau}^{\lambda} = F_{\lambda\tau} s^{\lambda}.$$

It is well known that the ponderomotoric four-force density introduced by Lorentz in the electron theory is given by

$$(6.5) \quad k_\tau \stackrel{\text{def}}{=} F_{\lambda\tau} s^{\lambda} \equiv \left\{ \frac{1}{c_0} (\mathbf{i}\mathfrak{E}), - \left[\varrho \mathfrak{E} + \frac{1}{c_0} (\mathbf{i} \times \mathfrak{B}) \right] \right\}.$$

In ponderable matter with ε and μ different from 1 it is not easy to find a unique expression for the force density acting on matter [21]. However, according to the above equation (6.4) it is obvious, that k_τ is the tensorial divergence of the tensor

$$(6.6) \quad T_{\tau}^{\lambda} = -\{F_{\tau\varrho} F^{\lambda\varrho} - \frac{1}{4} \delta_{\tau}^{\lambda} F_{\varrho\sigma} F^{\varrho\sigma}\}.$$

Based on elementary calculations for the pure spatial components of this tensor

$$T_{i.}^k = E_i D_k + B_i H_k - \frac{1}{2} \delta_{ik} (\mathfrak{E} \mathfrak{D} + \mathfrak{B} \mathfrak{H})$$

is obtained, being in the rest system identical with Maxwell's stress tensor in ponderable matter. Further,

$$\{T_{0.}^1, T_{0.}^2, T_{0.}^3\} = \mathfrak{E} \times \mathfrak{H} \equiv \frac{1}{c_0} \mathfrak{S},$$

where \mathfrak{S} is Poynting's vector, and

$$T_{0.}^0 = \frac{1}{2} (\mathfrak{E} \mathfrak{D} + \mathfrak{B} \mathfrak{H}) \equiv h.$$

In the rest system \mathfrak{S} and h are identical with the usually recognized expressions for the electromagnetic energy flux and energy density in stationary matter. Finally,

$$\{T_{1.}^0, T_{2.}^0, T_{3.}^0\} = -(\mathfrak{D} \times \mathfrak{B}) \equiv -c_0 g.$$

The form of equation (6.4) suggests that the left-hand side of (6.4) is the ponderomotive four-force density of the electromagnetic field k_τ and that $T_{\lambda.}^{\tau}$, as given by (6.6), represents the electromagnetic energy-momentum tensor. This would mean that the quantities \mathfrak{S} , h , g in every system of co-ordinates should be interpreted as the electromagnetic energy flux, energy density and momentum density, respectively. The above expressions for \mathfrak{S} , h and g are due to Minkowski; in vacuo they reduce to the corresponding expressions of the electron theory. This means, however, that the above equations (6.4) can be interpreted as the law of conservation of energy and momentum in homogeneous and isotropic ponderable matter.

Minkowski's electromagnetic energy-momentum tensor satisfies the same identity

$$(6.7) \quad T_{\tau.}^{\tau} \equiv 0,$$

as the tensor of the electron theory, but it is obvious that $T_{\tau.}^{\lambda} \neq T_{\lambda.}^{\tau}$. This «non-symmetry» of Minkowski's energy-momentum tensors has given rise to a long discussion in the literature and it was generally felt that this property

represented a real difficulty for Minkowski's theory [21]. Abraham, therefore, tried to construct a symmetrical expression for the electromagnetic energy-momentum tensor satisfying Planck's general relation

$$(6.8) \quad \mathfrak{g} = \frac{1}{c_0^2} \mathfrak{S},$$

in the rest system.

This difficulty, however, does not occur in the theory elaborated here. It is, namely, meaningless to ask for the symmetry properties of the mixed component of the energy-momentum tensor, but its pure covariant, as well as the pure contravariant components, are obviously symmetrical

$$T_{\alpha\beta} = g_{\beta\lambda} T_{\alpha}{}^{\lambda} = g_{\lambda\beta} T_{\alpha}{}^{\lambda} = g_{\lambda\alpha} T_{\beta}{}^{\lambda} = T_{\beta\alpha}$$

fulfilling Planck's relation (6.8).

Furthermore, Minkowski's energy-momentum tensor satisfies Laue's criterion of radiation [12]. This means, however, that the same energy-momentum tensor of the field fulfils both important physical criteria of the theory, if it is supposed that the electromagnetic field is induced in the introduced model space representing by its geometrical structure the dielectric and paramagnetic properties of the dielectricum.

Finally, a further problem should be dealt with briefly. Let a point-like test body with the rest mass m_0 and electric charge e be given, i.e.:

$$e = \int \varrho(\mathbf{r}') \delta_3(\mathbf{r}' - \mathbf{r}) d^3\mathbf{r}, \quad (d^3\mathbf{r} = dx^1 dx^2 dx^3),$$

$\delta_3(v)$ being the three-dimensional Dirac's δ . Hence, onwing to (6.5)

$$k^\tau = -F^{\tau\lambda} s_\lambda = \left\{ \frac{1}{\sqrt{\mu}} \frac{\varepsilon\mu}{c_0} (i\mathfrak{E}), \quad \frac{1}{\sqrt{\mu}} \left[\varrho\mathfrak{E} + \frac{1}{c_0} (i \times \mathfrak{B}) \right] \right\},$$

the equation of motion is given by

$$m_0 \frac{d^2\mathbf{r}}{d\tau^2} = \frac{e}{\sqrt{\mu}} \{ \mathfrak{E} + (i \times \mathfrak{B}) \},$$

with $s^\mu = \varrho(dx^\mu/ds)$, as is usual in the electron theory. Now, the presence of the factor $1/\sqrt{\mu}$ on the right-hand side of this equation should be justified. On the basis of the definition of the metrical fundamental form

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{\varepsilon\sqrt{\mu}} (dx^0)^2 - \sqrt{\mu} [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] = \\ &= \frac{c_0^2}{\varepsilon\sqrt{\mu}} dt^2 \left\{ 1 - \frac{\varepsilon\mu}{c_0^2} \left[\left(\frac{dx^1}{dt} \right)^2 + \left(\frac{dx^2}{dt} \right)^2 + \left(\frac{dx^3}{dt} \right)^2 \right] \right\} = \frac{c_0^2}{\varepsilon\sqrt{\mu}} d\tau^2, \end{aligned}$$

is obtained, where $d\tau$ denotes the invariant proper time interval. However, the usual relation between the infinitesimal distance and the proper time interval is given by

$$ds = \frac{c_0}{\sqrt{\epsilon\mu}} d\tau'.$$

Therefore, it is obvious that in our case the gauge of the time unit is reduced by the proportion $\sqrt[4]{\mu}$. If we want to use the usual definition of the proper time interval, it must taken into account that

$$d\tau' = \sqrt[4]{\mu} d\tau.$$

The reason of this abbreviation of the unit of the time intervals is due to the fact that in the case of dielectrics with $\mu \neq 1$, in order to derive the metrical fundamental tensor (5.1), the transformation of co-ordinates corresponding to (2.2) must be substituted by

$$x^0 = \sqrt{\epsilon\sqrt{\mu}} \cdot \xi^0, \quad x^k = (1/\sqrt[4]{\mu}) \xi^k.$$

Therefore, the correct equation of motion is given by

$$(6.9) \quad m_0 \frac{d^2 v}{d\tau'} = e \left\{ \mathfrak{E} + \frac{\sqrt{\epsilon\mu}}{c_0} \left(\frac{dv}{d\tau'} \times \mathfrak{B} \right) \right\},$$

as was expected.

7. - Discussions.

The programme of the above considerations was to elaborate the theory of the electromagnetic field according to the suggested new geometrical method in a quite heuristical way, in order to throw a strong light upon the general idea proposed. This elaboration of the concepts step by step was somewhat lengthy, however, it was necessary to make clear the physical and geometrical background of the method. Now, in this paragraph the invariant-theoretical basis of the elaborated theory will be briefly discussed too.

Let us consider the conformal transformation of the metric of the space

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = \sigma g_{\alpha\beta},$$

where $\sigma = \sigma(x)$ is a scalar. Taking the infinitesimal conformal transformation

$$\delta g^{\alpha\beta} = g'^{\alpha\beta} - g^{\alpha\beta} = g^{\alpha\beta} \delta \chi$$

and owing to (5.3), (6.3) and (6.7), for the variation of the Lagrangian

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} g^{\alpha\beta} \delta \chi = \frac{1}{2} \sqrt{|g|} T_{\alpha\beta} g^{\alpha\beta} \delta \chi \equiv 0,$$

is obtained. This means that the Lagrangian (5.3) is conformal-invariant [32].

Bearing this property of the electromagnetic field in mind, it becomes obvious that by the change of the metric of the space, where the field is induced, *i.e.* by the introduction of the suggested model space, the field equations and the whole formalism of the theory of the electromagnetic field will not be varied. This is the direct reason why the formalism of the theory of the field preserves its elegance being so suggestive in vacuum.

Finally, two conclusions should be drawn concerning the above considerations. Owing to the introduction of the model space, it becomes obvious that for the case of the dielectricum the exclusive natural generalization of the energy-momentum tensor of the electromagnetic field in vacuum is the Minkowskian one defined by equation (6.6). Furthermore, it follows from (6.7) that the important conclusion of the theory in vacuum, namely: the vanishing of the rest mass of photons, preserves its validity in dielectrics too, on supposing that the electric and magnetic property of the dielectrics is represented by the geometrical structure of the model space. This is a very remarkable conclusion of the theory being in contradiction with the opinion and results of previous authors [22]. It is, however, needless to say that this is an immediate consequence of the introduction of the light-metric in the model space.

Further conclusions of the elaborated theory, as well as the application of the method in the case of anisotropic dielectrics discussed above only in broad outlines, will be treated elsewhere.

II. — A model example of classical bilocal theory of physical fields.

8. — The basic idea of bilocal theory of fields.

The concept of bilocality of physical fields was introduced by Yukawa [38]. Since then, several authors have dealt in detail with this very interesting theory, particularly, from the point of view of the theory of elementary particles [27, 28].

The general idea of the physical field theory suggested originally by Faraday and Maxwell was that—instead of the concept of the point mechanics according to which the action of the forces is an action at distance—the interaction between two separate particles is transmitted by the physical field. This means that the change of the state of the field in a point of the space-time world depends only on the changing of the state of the stress of the field in

the immediate neighbourhood of the considered space-time point. If we take only the action of the state of the stress of the field in the infinitesimal neighbourhood of the considered space-time point into account some field equations can be deduced by the well known limiting process for the characterization of the balance, which are owing to the above considerations partial differential equations. The theory of the field is in this case an ordinary local field theory.

If the mentioned limiting process is, however, neglected it may be assumed that the state of the field in the considered space-time point is directly influenced by the phenomena taking place simultaneously in the space-time point of a sphere with a radius λ surrounding the considered point. Apparently this assumption represents the general idea of the non-local theory of physical fields. It is clear that according to this assumption the idea of the action at a distance recalled in the inside of the sphere with radius λ , however, from the field theoretical point of view this assumption does not represent any contradiction, because the aspect of the field theory remains macroscopically—between elementary particles—unchanged.

In Yukawa's bilocal theory the quantities of the field in their matrix representation depend on a point pair of the four-dimensional space-time continuum. In the corresponding classical theory of field the different quantities of the field have to be functions of these point pairs. Consequently the natural geometrical basis for the bilocal field theory is the space of point pairs $(x_{(i)}, x_{(j)})$. The ensemble of the point pairs of the world continuum should be called space \mathfrak{B} , the fundamental element of which is $(x_{(i)}, x_{(j)})$.

The point pair $(x_{(i)}, x_{(j)})$ is space-like, light-like and time-like, respectively, according to

$$(x_{(i)}^2 - x_{(j)}^0)^2 - (x_{(i)}^1 - x_{(j)}^1)^2 - (x_{(i)}^2 - x_{(j)}^2)^2 - (x_{(i)}^3 - x_{(j)}^3)^2 \leq 0.$$

Now, let us introduce the so-called co-ordinates of Yukawa by the definition

$$x^\mu = \frac{1}{2}(x_{(i)}^\mu + x_{(k)}^\mu), \quad r^\mu = x_{(i)}^\mu - x_{(k)}^\mu.$$

Instead of the point pairs $(x_{(i)}, x_{(k)})$ we can use the co-ordinates (x, r) . Let the ensemble of the co-ordinates (x, r) be regarded as the basic element of the \mathfrak{Y} -space. Since in Yukawa's theory, owing to physical arguments, for the vector r^μ : a condition of normalization is assumed by

$$(8.1) \quad r^\mu r_\mu = \lambda^2,$$

where λ is a constant, the four components of r^μ are not independent, consequently, only one direction is determined by the second group of the co-ordi-

nates of YUKAWA. This means, however, that *the basic element of the space \mathfrak{Y} can be regarded as a ground element of a line-element space*, the concept of which was introduced in Sect. 2.

The condition of normalization which is introduced in Yukawa's theory as an auxiliary condition can be substituted by an adequate one, as follows: *Let us introduce instead of Yukawa's co-ordinates r^μ the co-ordinates*

$$(8.2) \quad v^\mu = \varrho r^\mu,$$

where ϱ is an arbitrary positive factor and it should be supposed that the quantities of the physical field are homogeneous functions of the variable v^μ of zero degree. This means, however, that the quantities of the field depend on the basic element (x, v) of the line-element space \mathfrak{L} . Since only a direction is defined by the vector v^μ it is evident that the r^μ -s can be regarded as homogeneous directional co-ordinates, their proportion only having a meaning. So, in (8.2) the introduced factor ϱ has not any role in the following and the new denotation of the second group of Yukawa's co-ordinates was carried out in order to recall emphatically that the co-ordinates r^μ must fulfil the condition

$$v^\mu v_\mu = \text{const.},$$

because of which only three of them are independent.

However, owing to Euler's theorem the homogeneity of an arbitrary function of v^μ of zero degree means that

$$(8.3) \quad v^\mu f_{,(\mu)} = 0,$$

where the notation

$$(8.4) \quad f_{,(\mu)} \stackrel{\text{def}}{=} \frac{\partial f}{\partial v^\mu},$$

is introduced. (8.3) has the geometrical meaning that the derivative of a scalar of the field according to v^μ is a covariant vector being orthogonal to the direction of the basic line-element determined by v^μ .

Otherwise, the condition (8.1) is very characteristic for Yukawa's bilocal theory of fields, inasmuch as, the bilocal theory is just distinguished from the non-local theory of fields by the condition (8.1). Namely, also in the case of non-local theory of fields Yukawa's variables can be introduced and so the formulation of both theories seems to be quite adequate [20]. However, in the case of non-local fields the differences of the points $x_{(i)}^\mu$ and $x_{(k)}^\mu$ are not subjected to the restriction (8.1), this means that by $r^\mu = x_{(i)}^\mu - x_{(k)}^\mu$ not only a direction, but, directly the difference of $x_{(i)}^\mu$ and $x_{(k)}^\mu$, respectively, is deter-

mined; *i.e.* in the case of bilocal fields the field quantities depend on seven independent variables (four position co-ordinates x^μ and three independent directional co-ordinates), however, in the case of non-local fields the quantities of the field are functions of eight independent co-ordinates (four-position co-ordinates x^μ and four difference of co-ordinates r^μ). It is not doubtful that this circumstance is important from the point of view of the understanding of the basic concept of the bilocal theory not yet pointed out.

In order to take explicitly in a covariant way into account that the quantities of the field depend on three independent directional co-ordinates let us introduce three vectors $\hat{\lambda}_{(i)}^\mu$ ($i=1, 2, 3$) determining three arbitrary different directions. Then, the three angles

$$(8.5) \quad \vartheta_i = \arccos \{g_{\mu\nu} v^\mu \hat{\lambda}_{(i)}^\nu / [(g_{\alpha\sigma} v^\alpha v^\sigma)^{\frac{1}{2}} (g_{\alpha\beta} \hat{\lambda}_{(i)}^\alpha \hat{\lambda}_{(i)}^\beta)^{\frac{1}{2}}]\}$$

can be obtained (where $g_{\mu\nu}(x, v)$ is the metrical fundamental tensor of the line-element space \mathfrak{L}) which can be regarded as *inhomogeneous directional co-ordinates in all points of the space*. The angles ϑ_i were introduced in an invariant way. This means, however, that the angles ϑ_i are invariants of the general group of co-ordinate transformations.

It is obvious that by (8.5) a one to one correspondence between the proportion of r^μ -s and ϑ_i -s is determined.

Finally, it should be taken into account that in the case of line-element spaces, the derivatives of a field quantity in respect to the co-ordinates x^μ depended on the homogeneous directional co-ordinates. In order to restrict the possibilities, it may be demanded, corresponding to (8.4), that *the gradient of the field quantities has to be orthogonal to the direction of the line element, i.e.*

$$(8.6) \quad v^\mu \nabla_\mu f = 0,$$

where ∇_μ denotes the covariant derivative in the line-element space. If the covariance is only suggested against linear transformations of co-ordinates, the partial derivative in respect to the co-ordinates x^μ is a covariant vector too and (8.6) can be replaced by

$$(8.7) \quad v^\mu \partial_\mu f = 0.$$

Condition (8.7) corresponds to the well known auxiliary conditions of Yukawa's theory.

9. — Geometrical characterization of the space.

The ensemble of the fundamental elements (x, v) was called in the previous paragraph line-element space. The geometrical structure of the line-element space can be defined by the metrical fundamental tensor $g_{\mu\nu} = g_{\mu\nu}(x, v)$ which

can be determined arbitrarily and it should only be assumed that $g_{\mu\nu}$ are homogeneous functions of the variables v^μ of zero degree [10].

The line-elements (x, v) will be changed by transformation of the co-ordinates as follows:

$$x^{\mu'} = x^{\mu''}(x^\mu), \quad v^{\mu'} = \frac{\partial x^{\mu''}}{\partial x^\mu} v^\mu, \quad \Delta \stackrel{\text{def}}{=} \left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right| \neq 0,$$

and the law of transformation of tensors is given by

$$T^{\alpha'\beta'}_{\dots\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial x^{\gamma'}}{\partial x^{\gamma'}} T^{\alpha\beta}_{\dots\gamma}.$$

The one-parametric sequence of the line-elements

$$x^\mu = x^\mu(t), \quad v^\mu = v^\mu(t) \quad (t_1 \leq t \leq t_2)$$

is defined as a curve of space \mathfrak{L} for the direction field $v^\mu = v^\mu(t)$. The length of the arc of this curve is determined by

$$s = \int_{t_0}^{t_*} \left\{ g_{\mu\nu}(x, v) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right\}^{\frac{1}{2}} dt.$$

Now, let the scalar fundamental function

$$F(x, v) \stackrel{\text{def}}{=} \{ g_{\mu\nu}(x, v) v^\mu v^\nu \}^{\frac{1}{2}}$$

be introduced being a homogeneous function of the variable v^μ of first degree. With help of $F(x, v)$ the vector of unit length

$$(9.1) \quad l^\mu \stackrel{\text{def}}{=} v^\mu / F$$

can be defined in the direction of the line-element. Owing to (9.1) the inhomogeneous directional co-ordinate of the line-element, introduced by (8.5), is given by

$$(9.2) \quad \vartheta_i = \arccos \{ g_{\mu\nu}(x, l) l^\mu l^\nu / (g_{\alpha\beta}(x, l) l^\alpha l^\beta)^{\frac{1}{2}} \}.$$

In our space the invariant differential of a vector ξ^μ is defined by

$$D\xi^\mu = d\xi^\mu + C^\mu_{\alpha\lambda} \xi^\alpha dv^\lambda + \Gamma^\mu_{\alpha\lambda} \xi^\alpha dx^\lambda,$$

where $C^\mu_{\alpha\lambda}$ and $\Gamma^\mu_{\alpha\lambda}$ are the so-called components of connection of the space, which can be calculated if $g_{\mu\nu}$ is known [10].

The parallel displacements of vectors in the sense of Levi-Civita are defined by

$$D\xi^\mu = 0.$$

The covariant derivative of tensors is given by

$$\nabla_\lambda T_{\alpha\gamma}^\beta = T_{\alpha\gamma,\lambda}^\beta - T_{\alpha\gamma,(\varrho)}^\beta \Gamma_{0\lambda}^{*\varrho} - \Gamma_{\alpha\lambda}^{*\varrho} T_{\varrho\gamma}^\beta + \Gamma_{\varrho\lambda}^{*\beta} T_{\alpha\gamma}^\varrho - \Gamma_{\gamma\lambda}^{*\varrho} T_{\alpha\varrho}^\beta,$$

where the abbreviations

$$\Gamma_{\alpha\lambda}^{*\varrho} \stackrel{\text{def}}{=} \Gamma_{\alpha\lambda}^\varrho - A_{\alpha\tau}^\varrho \Gamma_{0\lambda}^\tau; \quad A_{\alpha\tau}^\varrho \stackrel{\text{def}}{=} FC_{\alpha\tau}^\varrho$$

are introduced and the index « 0 » means contraction with the vector l^μ , e.g.:

$$T_{\sigma\gamma}^{\lambda} \stackrel{\text{def}}{=} T_{\mu\gamma}^{\lambda} l^\mu; \quad \Gamma_{0\lambda}^{*\varrho} = \Gamma_{\alpha\lambda}^{*\varrho} l^\alpha.$$

The equation of the *extremal curves*, or the *geodetical lines of the space* are:

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma_{\mu\gamma}^{*\lambda} \frac{dx^\mu}{ds} \frac{dx^\gamma}{ds} = 0,$$

being the Euler-Lagrange equation of the variational principle

$$\delta \int_{s_1}^{s_2} \left\{ g_{\mu\nu}(x, v) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right\}^{\frac{1}{2}} ds = 0.$$

The curvature of the space is given by the tensors

$$\begin{aligned} R_{\kappa\varrho\tau}^\mu &= R_{\kappa\varrho\tau}^{*\mu} + A_{\kappa\sigma}^\mu R_{0\varrho\tau}^{*\sigma}, \\ P_{\kappa\varrho\tau}^\mu &= F\Gamma_{\kappa\varrho,(\tau)}^{*\mu} - \nabla_\varrho A_{\kappa\tau}^\mu + A_{\kappa\sigma}^\mu \Gamma_{\nu\varrho,(\tau)}^{*\sigma} l^\nu, \\ S_{\kappa\varrho\tau}^\mu &= 2A_{\kappa[\tau}^\sigma A_{\varrho]\sigma}^\mu, \end{aligned}$$

where

$$(9.3) \quad R_{\kappa\varrho\tau}^{*\mu} \stackrel{\text{def}}{=} 2\Gamma_{\kappa[\varrho,\tau]}^{*\mu} + 2F\Gamma_{0[\varrho}^{*\sigma} \Gamma_{\tau]\kappa,(\sigma)}^{*\mu} - \Gamma_{\sigma[\varrho}^{*\mu} \Gamma_{\tau]\kappa}^{*\sigma}$$

is the tensor of *principal curvature* and $T_{[\mu] \tau [\nu]}$ is generally an abbreviation introduced by SCHOUTEN

$$T_{[\mu] \tau [\nu]} \stackrel{\text{def}}{=} \frac{1}{2} \{ T_{\mu\tau\nu} - T_{\nu\tau\mu} \}.$$

By contraction of the indices μ and τ of $R_{\kappa\varrho\tau}^\mu$ *Einstein-Ricci's tensor*

$$(9.4) \quad R_{\kappa\varrho} \stackrel{\text{def}}{=} R_{\kappa\varrho\mu}^\mu,$$

furthermore, by contraction with $g^{\kappa\varrho}$ the scalar curvature of the space

$$(9.5) \quad R \stackrel{\text{def}}{=} g^{\kappa\varrho} R_{\kappa\varrho}$$

can be derived.

It was mentioned in Sect. 2, that a homogeneous and anisotropic space is characterized by the metrical fundamental tensor $g_{\mu\nu} = g_{\mu\nu}(v)$. Now, let us suggest that the geometrical basis of the bilocal field is a homogeneous and anisotropic space, i.e. its metrical fundamental tensor depends only on the directions v^μ being a so-called Minkowskian space [34]. In this case there exists a frame of reference in which $\Gamma_{\alpha,\gamma}^{*\beta} = 0$; that is, the scalar curvature of the space vanishes. Therefore, the space is a flat one.

In classical physics the field is described by one or several (real) space-time functions $\psi_\mu = \psi_\mu(x)$ which satisfy certain partial differential equations, the so-called field equations. A usual alternative procedure is to start with a variational principle chosen in such a way that its Euler-Lagrange differential equations are identical with the field equations. This method renders possible the so-called canonical formalism of the field.

The canonical formalism of the theory starts with the definition of the Lagrangian \mathfrak{L} of the field. In the usual point space the Lagrangian is a scalar density and therefore the volume integral

$$I = \int_{\Omega} \mathfrak{L}(x) d^4x,$$

—the so-called *integral of action*—is an invariant of the transformation of co-ordinates.

The scalar and tensor densities, respectively, can also be defined in the space of line-elements by the usual law of transformation:

$$\mathfrak{L}' = \Delta^{-1} \mathfrak{L}; \quad \mathfrak{T}_{\alpha',\gamma'}^{\beta'} = \Delta^{-1} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\gamma}{\partial x^{\gamma'}} T_{\alpha,\gamma}^\beta,$$

respectively, and the Lagrangian $\mathfrak{L} = \mathfrak{L}(x, v)$ can be introduced without any difficulties. However, the integral of action

$$(9.6) \quad I(v) = \int_{\Omega} \mathfrak{L}(x, v) d^4x,$$

depending on v , has only a definite meaning in the usual sense if, in all points of the space a direction is given by $v'' = v''(x)$. In this case—as one says—the integral of action (9.6) refers to the field direction $v'' = v''(x)$.

In the following we shall not deal with the definition quite generally which is associated with unnecessary difficulties, but only with a special case which seems to be sufficient for our purpose:

Let a field of directions be $v'' = v''(x)$ satisfying the differential equation

$$(9.7) \quad \frac{dl^\mu}{dx^e} + \Gamma_{0 \cdot e}^{*\mu} = 0,$$

where

$$\frac{dl^\mu}{dx^e} \equiv \partial_e l^\mu(x, v(x)).$$

The condition of integrability of the differential equation (9.7) is

$$(9.8) \quad \partial_e \partial_\tau l^\mu - \partial_\tau \partial_e l^\mu \equiv -R_{0 \cdot e \tau}^{*\mu} = 0.$$

On the basis of a theorem of Frobenius [33] the fulfilment of equation (9.8) means that *in our space of line-elements there exists a parallel displacement of line-elements*. This is a restricting condition for the space with the immediate geometrical meaning that for a given direction in an arbitrary space-time point in every other point of our space-time world a parallel direction in the sense of Levi-Civita can be determined uniquely.

Therefore, in the following the integral of action (9.6) in respect to the field of direction $v'' = v''(x)$ fulfilling equation (9.7) will be defined.

In the case of Minkowskian spaces the tensor of the principal curvature, $R_{\mu \cdot \nu \sigma}^e$ vanishes identically, therefore the condition of integrability of equation (9.7) is fulfilled and the integral of action (9.6) can be introduced without difficulties.

10. - Generalized Schrödinger-Gordon equation of the bilocal fields.

In the following the scalar bilocal field theory will be dealt with. In this case the field is characterized by a scalar bilocal function $\psi = \psi(x, v)$. The type of this theory of fields—owing to the terms introduced above—is a field theory of second kind. Let the Minkowskian space with the metrical fundamental tensor $g_{\alpha\beta} = g_{\alpha\beta}(v)$ be the geometrical background of the theory. The explicit dependence of the metrical fundamental tensor on the homogeneous directional co-ordinates will not be fixed.

Let

$$(10.1) \quad v^\alpha = v^\alpha(x)$$

be a field of directions fulfilling our previous equations (9.7). Such a field of directions exists because the space is a Minkowskian one, where by the vanishing

of the principal curvature tensor of the space the condition of integrability of the differential equation (9.7) is automatically fulfilled. Then the integral of action of the field is given by

$$I(v) \equiv \int_{\Omega} \mathfrak{L}(x, v) d^4x,$$

where Ω is the four-dimensional domain of integration and the Lagrangian density of the field is a known function of the contravariant component of the metrical fundamental tensor $g^{\alpha\beta}$, as well as that of the scalar field component ψ and its derivatives:

$$\mathfrak{L} = \sqrt{|g|} \cdot L[g^{\alpha\beta}(x, v), \psi(x, v), \psi_{,\alpha}(x, v)].$$

Varying the function ψ for the fixed region of integration, subjected to the restriction that the variation of ψ and its first derivatives at the boundary of the domain of integration vanish, one obtains by partial integration in the familiar way [10]:

$$\delta I \equiv \int_{\Omega} \left\{ \frac{\partial \mathfrak{L}}{\partial \psi} - \partial_{\alpha} \frac{\partial \mathfrak{L}}{\partial \psi_{,\alpha}} \right\} \delta \psi d^4x.$$

Now, for an arbitrary variation of ψ satisfying the above mentioned conditions and for arbitrary choice of the integration region the classical field must be determined by the condition that the integral I should be stationary, *i.e.*

$$\delta I \equiv 0.$$

From this follows in all space-time points:

$$(10.2) \quad \frac{\partial \mathfrak{L}}{\partial \psi} - \partial_{\alpha} \frac{\partial \mathfrak{L}}{\partial \psi_{,\alpha}} = 0, \quad \text{for } v^{\alpha} = v^{\alpha}(x).$$

This differential equation is *the field equation of the scalar field in its implicit form*. In order to determine the explicit dependence of ψ on x^{μ} , as well as on r^{μ} we should introduce instead of the homogeneous v^{μ} -s the inhomogeneous directional co-ordinates ϑ_i -s which, on the basis of the absolute parallelism of line-elements of space are uniquely determined in all space-time points of the Minkowskian space:

$$I(\vartheta_i) \equiv \int_{\Omega} \mathfrak{L}(x, \vartheta_i) d^4x.$$

Owing to the invariance of ϑ_i -s against any transformation of point co-ordinates this integral of action is itself an invariant too. In order to determine the dependence of the field function ψ on ϑ_i too let us introduce the new integral of action:

$$(10.3) \quad I^* \stackrel{\text{def}}{=} \int I(\vartheta) d^3\vartheta = \iint_{\Omega} \mathfrak{L}(x, \vartheta) d^4x d^3\vartheta,$$

and assume that for arbitrary variation of ψ , again subjected to the restriction that the variation of ψ and its derivatives according to x'' , as well as to ϑ_i at the boundary of the domain of integration vanish, the classical field is determined by the condition that the integral I^* must be stationary, *i.e.*

$$(10.4) \quad \delta I^* \equiv 0.$$

Let the implicate dependence of the Lagrangian on the field function ψ , as well as on its partial derivatives be determined by

$$\mathfrak{L} = \sqrt{|g|} L(\psi, \psi_{,\alpha}, \psi_{,(i)}) ,$$

then by partial integrations

$$\begin{aligned} \delta^* I &= \iint_{\Omega} \left\{ \frac{\partial \mathfrak{L}}{\partial \psi} \delta \psi + \frac{\partial \mathfrak{L}}{\partial \psi_{,\alpha}} \delta \psi_{,\alpha} + \frac{\partial \mathfrak{L}}{\partial \psi_{,(i)}} \delta \psi_{,(i)} \right\} d^4x d^3\vartheta = \\ &= \iint_{\Omega} \left\{ \frac{\partial \mathfrak{L}}{\partial \psi} - \partial_{\alpha} \frac{\partial \mathfrak{L}}{\partial \psi_{,\alpha}} - \partial_{(i)} \frac{\partial \mathfrak{L}}{\partial \psi_{,(i)}} \right\} \delta \psi d^4x d^3\vartheta, \end{aligned}$$

is obtained, where the notation

$$(10.5) \quad \psi_{,(i)} \equiv \partial_{(i)} \psi \stackrel{\text{def}}{=} \frac{\partial \psi}{\partial \vartheta_i},$$

is introduced. Furthermore, owing to (10.4) and taking the arbitrariness of $\delta \psi$ into account, the field equation

$$(10.6) \quad \frac{\partial \mathfrak{L}}{\partial \psi} - \partial_{\alpha} \frac{\partial \mathfrak{L}}{\partial \psi_{,\alpha}} - \partial_{(i)} \frac{\partial \mathfrak{L}}{\partial \psi_{,(i)}} = 0,$$

is obtained.

Finally, let us suppose that the Lagrangian of the field is quadratic in ψ and its derivatives, that is, *e.g.*:

$$(10.7) \quad \mathfrak{L} = \frac{1}{2} \sqrt{|g|} \{ A^{\mu\nu} \psi_{,\mu} \psi_{,\nu} + B^{ik} \psi_{,(i)} \psi_{,(k)} + C^{\alpha} \psi \psi_{,\alpha} + D^i \psi \psi_{,(i)} + E \psi^2 \},$$

where the coefficients $A^{\mu\nu}$, B^{ik} , C^α , D^i and E , respectively, are constants, or functions of ϑ_i -s fulfilling the symmetry relations:

$$(10.8) \quad A^{\mu\nu} = A^{\nu\mu}, \quad B^{ik} = B^{ki}.$$

Consequently the field equations

$$(10.9) \quad \{A^{\alpha\beta} \partial_\alpha \partial_\beta + B^{ik} \partial_{(i)} \partial_{(k)} - E\} \psi(x, \vartheta) = 0$$

or

$$(10.10) \quad \{A^{\alpha\beta}(\vartheta) \partial_\alpha \partial_\beta + B^{ik}(\vartheta) \partial_{(i)} \partial_{(k)} + B_{..(i)}^{ik}(\vartheta) \partial_{(k)} + \frac{1}{2} D_{..(i)}^i(\vartheta) + E(\vartheta)\} \psi(x, \vartheta) = 0$$

are obtained.

Let us suppose that

$$(10.11) \quad \psi(x, \vartheta) = X(x) \theta(\vartheta),$$

then the field equations are separable and

$$(10.12) \quad A^{\alpha\beta} X_{,\alpha\beta} - \mu^2 X = 0; \quad B^{ik} \theta_{,(ik)} + (\mu^2 - E) \theta = 0,$$

as well as

$$(10.13) \quad \begin{cases} A^{\alpha\beta}(\vartheta) X_{,\alpha\beta} - \mu^2 X = 0, \\ B^{ik}(\vartheta) \theta_{,(ik)} + B_{..(i)}^{ik}(\vartheta) \theta_{,(k)} + (\mu^2 + \frac{1}{2} D_{..(i)}^i(\vartheta) - E(\vartheta)) \theta = 0 \end{cases}$$

are obtained.

The constant of separation μ is generally quantized and can be brought into relation with the mass quantization.

If

$$(10.14) \quad A^{\alpha\beta} = g^{\alpha\beta}(\vartheta), \quad B^{ik} = g^{ik}(\vartheta), \quad D^i = \text{const}, \quad E = \kappa^2$$

and the notations

$$(10.15) \quad \square_{(x)} \stackrel{\text{def}}{=} g^{\alpha\beta} \partial_\alpha \partial_\beta \quad \text{and} \quad \Delta_{(\vartheta)} \stackrel{\text{def}}{=} \partial_{(i)} (g^{ik} \partial_{(k)})$$

are introduced the field equation

$$(10.16) \quad \{\square_{(x)} + \Delta_{(\vartheta)} - \kappa^2\} \psi(x, \vartheta) = 0$$

is obtained being the *generalized Schrödinger-Gordon equation* of the bilocal field.

Let the original homogeneous directional co-ordinates v^α again be intro-

duced, then

$$\Delta_{(v)} \equiv \frac{\partial v_\alpha}{\partial \vartheta_i} \frac{\partial}{\partial v^\alpha} \left(g^{ik} \frac{\partial v^\beta}{\partial \vartheta_k} \frac{\partial}{\partial v^\beta} \right),$$

and for the field equation explicitly

$$(10.17) \quad g^{\alpha\beta} \frac{\partial^2 \psi(x, v)}{\partial x^\alpha \partial x^\beta} + \frac{\partial v^\alpha}{\partial \vartheta_i} \frac{\partial}{\partial v^\alpha} \left(g^{ik} \frac{\partial v^\beta}{\partial \vartheta_k} \frac{\partial \psi(x, v)}{\partial v^\beta} \right) - \kappa^2 \psi(x, v) = 0,$$

is obtained which is the explicit form of the generalized Schrödinger-Gordon equation. By substituting the factorized form of

$$\psi(x, v) = X(x)V(v)$$

into (10.17) we have instead of (10.13)

$$(10.18) \quad g^{\alpha\beta} \frac{\partial^2 X}{\partial x^\alpha \partial x^\beta} - \mu^2 X = 0, \quad \frac{\partial v^\alpha}{\partial \vartheta_i} \frac{\partial}{\partial v^\alpha} \left(g^{ik} \frac{\partial v^\beta}{\partial \vartheta_k} \frac{\partial V}{\partial v^\beta} \right) + (\mu^2 - \kappa^2) V = 0,$$

being the separated field equations of the bilocal field.

11. - Discussions.

It is well known that the second group of Yukawa's variables v^α , as well as the corresponding inhomogeneous co-ordinates of direction ϑ_i , respectively, are usually interpreted as the co-ordinates of internal motions of the field. This introduced internal degree of freedom is related to the non-localizability of the field itself. The operators

$$F_{\text{op}}(v) \stackrel{\text{def}}{=} \Delta_{(v)} + E(v), \quad \text{or} \quad F_{\text{op}}(\vartheta) \stackrel{\text{def}}{=} \Delta_{(\vartheta)} + E(\vartheta)$$

can be chosen in such a way that their spectrum should be positive and discrete the eigenvalues μ_n ($n=1, 2, 3, \dots$) being the masses of the different elementary particles the family of which is generated by the theory. The eigenfunctions $\theta_n(\vartheta)$ and $V_n(v)$, respectively, are usually termed internal eigenfunctions. The internal eigenfunctions play the role of the convergence factor of the non-local fields, however, as was pointed out by Yukawa there is an essential difference between the bilocal and the non-local theory of fields. Namely, in the case of the bilocal theory of fields all the particles μ_n which were derived from the eigenvalue problems (10.10) or (10.17) are taken simultaneously into account. Furthermore, the form functions for each of these particles are uniquely de-

terminated by the same eigenvalue problem. However, further details of the theory of elementary particles will not be dealt with here.

The equations (10.10) and (10.17), as well as (10.13) and (10.18), respectively, are completely equivalent. Often the use of (10.10) and (10.13) is more suitable. Otherwise, they are equivalent with the equations of Yukawa and in a particular case they reduce to Rayski's ones.

Now, let us take the three-dimensional hyperplane

$$(11.1) \quad x^0 = \text{const},$$

into account, then in all points of the hyperplane the homogeneous directional co-ordinates v'' are distinguished by $v^0 = 0$. These directional co-ordinates are, however, subjected to the restriction $v'' v_\mu = \text{const}$, therefore, in this case only two independent inhomogeneous directional co-ordinates can be introduced. (This circumstance corresponds to the fact that in the three-dimensional space there are two independent polar angles). This means, however, that at contemporary points of the space-time world the internal structure of the bilocal field is determined by two internal co-ordinates leading to the rotator model of elementary particles of RAYSKI [26, 28].

The procedure mentioned is not independent of the frame of reference and is not covariant against Lorentz-transformations. However, if instead of the plane (11.1) a space-like hyperplane σ is introduced the above considerations can be given in a covariant way too.

III. – Differential structure of classical fields in generalized metrical spaces.

12. – Classical theory of fields of second kind in general spaces.

The above considerations may already illustrate the usefulness of the general concept of the new geometrical aspect suggested. The theory of electromagnetic field in moving dielectrics, as well as the classical theory of scalar bilocal field elaborated in the previous chapters in broad outlines do not claim the right of being a final and complete theory, however, they justified the fact that the elaboration of the theory of physical fields of second kind in generalized metrical spaces is from the physical point of view of interest for future investigations in the topic of theory of fields.

By the last analysis, as is obvious owing to the above considerations, the original Riemannian idea is recalled according to which the geometrical structure of the space is determined by the physical properties of the matter. As is well known, this Riemannian aspect was used by Einstein for a direct geometrization of the gravitational field in the case of the general theory of rela-

tivity. According to the new aspect, suggested here, the Riemannian idea and the original field-theoretical aspect should be reconciled by the assumption that the physical properties of the medium where the physical field is excited may be taken into account by the geometrical structure of the space and the field is described by the usual methods of the field theories of second kind having as geometrical background the adequate metrical space introduced.

In the case of the above first model example the metrical structure of the space was directly determined by the material constants of the medium. However, in the case of the bilocal field theory the natural geometrical basis seems directly to be a space with generalized metrical properties.

In this chapter the general theory will be dealt with. Namely, the metrical geometrical space in which the field is excited should be regarded as a generalized one, the structure of which is determined by an arbitrary metrical fundamental tensor.

For the sake of generality let us suppose that the space considered is a generalized line-element space with the metrical fundamental tensor $g_{\mu\nu} = g_{\mu\nu}(x, v)$.

By defining the Lagrangian of the field in the introduced general space too, the covariant field equations and conservation laws will be deduced. The tensor of energy and impulse will be also derived on the basis of the results mentioned above in a quite general form, for the different well known cases elaborated previously by HILBERT [6], BORN [3], ROSENFELD [30] and DE WETT [5], respectively, it can be obtained directly by specialization.

13. — Geometrical preliminaries.

The geometrical preliminaries were discussed above, especially, in Sect. 9; further details can be found in previous paper [10]. In the present paragraph only some further concepts—important in the following—will be reviewed.

It is well known that in the immediate surroundings of a point of the Riemannian space a pseudo-Euclidian metric can be introduced; that is, to every point of the Riemannian space a «tangential pseudo-Euclidian space» can be given.

If in the space of line-elements absolute parallelism of the line-elements exists we can construct to every line-element of our space an osculating Riemannian space fulfilling the following conditions:

a) *The metrical fundamental tensor $\gamma_{\mu\nu}$ of the osculating Riemannian space is identical with the metrical fundamental tensor of the original line-element space, that is*

$$\gamma_{\mu\nu}(x) = g_{\mu\nu}(x, v(x)) ,$$

respectively, or owing to the homogeneity of zero order of $g_{\mu\nu}(x, v)$ in the

variable v'' :

$$(13.1) \quad \gamma_{\mu\nu}(x) = g_{\mu\nu}(x, l(x)) ;$$

b) *The geodetical lines of both spaces osculate each other;*

c) *The invariant differential and the covariant derivative of the vectors ξ'' are identical in both spaces;*

d) *The tensors of principal curvature of both spaces are the same.*

This construction of the osculating Riemannian space differs essentially from the Vargaian one [33] being far simpler and it is based on the existence of absolute parallelism in the space.

To prove the correctness of our construction we have to calculate the parameter of connection in the osculating Riemannian space:

$$\begin{aligned} \bar{\Gamma}_{\alpha\tau\beta} &\stackrel{\text{def}}{=} \frac{1}{2} \{ \partial_\beta \gamma_{\alpha\tau} + \partial_\alpha \gamma_{\beta\tau} - \partial_\tau \gamma_{\alpha\beta} \} = \\ &= \frac{1}{2} \{ \partial_\beta g_{\alpha\tau} + \partial_\alpha g_{\beta\tau} - \partial_\tau g_{\alpha\beta} \} + \frac{1}{2} \left\{ \frac{\partial g_{\alpha\tau}}{\partial l^\sigma} \partial_\beta l^\sigma + \frac{\partial g_{\beta\tau}}{\partial l^\sigma} \partial_\alpha l^\sigma - \frac{\partial g_{\alpha\beta}}{\partial l^\sigma} \partial_\tau l^\sigma \right\}. \end{aligned}$$

Based on (9.7) and taking into account that—owing to the homogeneity of (-1) -th order in l^q —

$$\frac{\partial g_{\alpha\beta}}{\partial l^\sigma} = F \partial_{(\sigma)} g_{\alpha\beta},$$

we obtain immediately

$$(13.2) \quad \bar{\Gamma}_{\alpha\tau\beta} = \Gamma_{\alpha\tau\beta}^*(x, l(x)).$$

Furthermore,

$$\partial_\tau \bar{\Gamma}_{\alpha\cdot\gamma}^\beta = \partial_\tau \Gamma_{\alpha\cdot\gamma}^{*\beta} - (\partial_{(\partial)} \Gamma_{\alpha\cdot\gamma}^{*\beta}) \Gamma_{0\cdot\tau}^{*\beta}$$

and based on the definition of the tensor of principal curvature

$$\bar{R}_{\alpha\cdot\varrho\tau}^\beta = R_{\alpha\cdot\varrho\tau}^{*\beta}(x, l(x)),$$

where $R_{\alpha\cdot\varrho\tau}^\beta$ is Riemann's tensor of curvature of the osculating Riemannian space.

However, in our case in the line-elements space absolute parallelism of the line-elements exists, hence

$$R_{\alpha\cdot\varrho\tau}^\alpha = R_{\sigma\cdot\varrho\tau}^{*\beta},$$

therefore

$$\bar{R}_{\alpha\cdot\varrho\tau}^\beta = R_{\alpha\cdot\varrho\tau}^\beta.$$

Q.e.d.

It is, of course, obvious that the construction of the osculating Riemannian space is independent of the particular choice of the co-ordinate system.

The infinitesimal transformation of the co-ordinates is defined also in the space of line-elements by

$$(13.3) \quad x^{\mu'} = x^{\mu} + \varepsilon \xi^{\mu}(x),$$

where ε is an infinitesimal parameter and $\xi^{\mu}(x)$ is an arbitrary covariant vector which is continuous and a limited function of the co-ordinates x .

Let $T(x, v)$ be a quantity of the space having an arbitrary law of transformation then we define its *total*, and *local variation*, respectively, as follows:

$$\delta T = T'(x', v') - T(x, v),$$

$$\delta^* T = T'(x, v) - T(x, v).$$

Considering that owing to the law of transformation of v^{μ}

$$(13.4) \quad \delta v^{\mu} = \varepsilon (\partial_{\rho} \xi^{\mu}) v^{\rho} + O(\varepsilon^2),$$

the connection between the two types of variation is given by

$$(13.5) \quad \delta T = \delta^* T + \varepsilon \{ (\partial_{\mu} T) \xi^{\mu} + (\partial_{(\mu} T) (\partial_{\rho} \xi^{\mu}) v^{\rho} \} + O(\varepsilon^2).$$

In the case of the local variation the operations δ^* and ∂_{μ} and $\partial_{(\rho}$, respectively, can be exchanged, that is,

$$\delta^* (\partial_{\mu} T) \equiv \partial_{\mu} (\delta^* T) \quad \text{and} \quad \delta^* (\partial_{(\rho} T) = \partial_{(\rho} (\delta^* T),$$

but in the case of the total variation

$$(13.6) \quad \delta (\partial_{\mu} T) = \partial_{\mu} (\delta T) - \varepsilon \{ (\partial_{\rho} T) \xi^{\mu} (\partial_{\mu} \xi^{\rho}) + (\partial_{(\rho} T) (\partial_{\mu} \partial_{\lambda} \xi^{\rho}) v^{\lambda} \} + O(\varepsilon^2)$$

and

$$(13.7) \quad \delta (\partial_{(\rho} T) = \partial_{(\rho} (\delta T) - \varepsilon (\partial_{(\rho} T) (\partial_{\rho} \xi^{\sigma}) + O(\varepsilon^2).$$

If the spatial law of transformation of T is given, it is possible—basing on our above results—to calculate the total and local variation explicitly; *e.g.* if T is a covariant tensor of second order

$$\delta T^{\mu\nu} = \varepsilon \{ (\partial_{\lambda} \xi^{\mu}) T^{\lambda\nu} + (\partial_{\lambda} \xi^{\nu}) T^{\mu\lambda} \} + O(\varepsilon^2)$$

and owing to (13.5)

$$\delta^* T^{\mu\nu} = -\varepsilon \{ (\partial_\lambda T^{\mu\nu}) \xi^\lambda - (\partial_\lambda \xi^\mu) T^{\lambda\nu} - (\partial_\lambda \xi^\nu) T^{\mu\lambda} + (\partial_{(\lambda} T^{\mu\nu})(\partial_{\rho)} \xi^\lambda) v^\rho \} + O(\varepsilon^2).$$

In the case of tensor densities we have to calculate the variation of $\sqrt{|g|}$, where

$$g \stackrel{\text{def}}{=} \det |g_{\mu\nu}|.$$

Since, as it is well known that

$$\delta \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu} = \frac{1}{2} \sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu},$$

we have

$$\delta \sqrt{|g|} = -\varepsilon \sqrt{|g|} (\partial_\lambda \xi^\lambda) + O(\varepsilon^2),$$

Now, for a tensor density

$$\delta \mathfrak{T} = T \cdot \delta \sqrt{|g|} + \sqrt{|g|} \cdot \delta T, \quad \delta^* \mathfrak{T} = \delta^* \sqrt{|g|} \cdot T + \sqrt{|g|} \delta^* T,$$

and *e.g.*

$$(13.8) \quad \delta^* g^{\mu\nu} = \varepsilon \sqrt{|g|} \{ (\partial_\lambda g^{\mu\nu}) \xi^\lambda - (\partial_\lambda \xi^\mu) g^{\lambda\nu} - (\partial_\lambda \xi^\nu) g^{\mu\lambda} + A^{\mu\nu}_{\cdot\lambda} (\partial_\tau \xi^\lambda) v^\tau \} + O(\varepsilon^2).$$

14. - Deduction of the field equations.

Let us consider a field of directions

$$(14.1) \quad v^\mu = v^\mu(x)$$

fulfilling our previous equation (9.7), then the integral of action of the field is given by

$$(14.2) \quad I(v) = \int_{\Omega} \mathfrak{L}(x, v(x)) \, d^4x,$$

where Ω is the four-dimensional domain of integration and the Lagrangian density of the field is a known function of the metrical ground tensor $g^{\mu\nu}$, as well as that of the components Φ_μ of the field and their derivatives:

$$\mathfrak{L} = \mathfrak{L}[g^{\mu\nu}(x, v(x)); \Phi_\mu(x, v(x)); \Phi_{\mu|\nu}(x, v(x)); \Phi_{\mu|\nu\lambda}(x, v(x))]$$

with

$$\Phi_{\mu|\nu} \stackrel{\text{def}}{=} \nabla_\nu \Phi_\mu \equiv \frac{d\Phi_\mu}{dx^\nu} - \Gamma^{\sigma\rho}_{\mu\nu} \Phi_\sigma,$$

and

$$\frac{d\Phi_{\mu}}{dx^{\nu}} \stackrel{\text{def}}{=} \partial_{\nu} \Phi_{\mu} - F(\partial_{(\nu} \Phi_{\mu)}) I_{0 \cdot \nu}^{\ast \varrho},$$

being the so-called Berwald's derivative of Φ_{μ} .

Varying the functions Φ_{μ} for the fixed region Ω of integration subjected to the restrictions that the variations of the Φ_{μ} -s and their first derivatives at the boundary of the domain of integration vanish, one obtains

$$\delta I = \int_{\Omega} \left\{ \frac{d\mathfrak{L}}{\partial \Phi_{\mu}} \delta \Phi + \left[\frac{\partial \mathfrak{L}}{\partial \Phi_{\mu|\nu}} - \frac{\partial \mathfrak{L}}{\partial \Phi_{\alpha|\nu\lambda}} \Gamma_{\alpha \cdot \lambda}^{\ast \mu} - \frac{\partial \mathfrak{L}}{\partial \Phi_{\mu|\alpha\lambda}} \Gamma_{\alpha \cdot \lambda}^{\ast \nu} \right] \delta \Phi_{\mu|\nu} + \frac{\partial \mathfrak{L}}{\partial \Phi_{\mu|\nu\lambda}} \frac{d}{dx^{\lambda}} \Phi_{\mu|\nu} \right\} d^4 x.$$

Based on the conditions of stationariness

$$\delta I \equiv 0,$$

we have by repeated partial integration

$$\int_{\Omega} \left\{ \frac{\partial \mathfrak{L}}{\partial \Phi_{\mu}} - \theta^{\nu} \Gamma_{\varrho \cdot \nu}^{\ast \mu} - \frac{d}{dx^{\nu}} \theta^{\mu\nu} \right\} \delta \Phi_{\mu} d^4 x \equiv 0,$$

where

$$(14.3) \quad \theta^{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \mathfrak{L}}{\partial \Phi_{\mu|\nu}} - \frac{\partial \mathfrak{L}}{\partial \Phi_{\alpha|\nu\lambda}} \Gamma_{\alpha \cdot \lambda}^{\ast \mu} - \frac{\partial \mathfrak{L}}{\partial \Phi_{\mu|\alpha\lambda}} \Gamma_{\alpha \cdot \lambda}^{\ast \nu} - \frac{d}{dx^{\lambda}} \frac{\partial \mathfrak{L}}{\partial \Phi_{\mu|\nu\lambda}}.$$

However, this equation is fulfilled for arbitrary variations of the Φ_{μ} -s which satisfy the above mentioned conditions and for an arbitrary choice of the integration region. Consequently for all space-time points

$$(14.4) \quad \frac{\partial \mathfrak{L}}{\partial \Phi_{\mu}} - \theta^{\nu} \Gamma_{\varrho \cdot \nu}^{\ast \mu} - \frac{d}{dx^{\nu}} \theta^{\mu\nu} = 0,$$

is obtained.

To put the field equations in their explicit covariant form we write $\theta^{\mu\nu}$ defined in (14.3) as

$$\theta^{\mu\nu} = \mathfrak{F}^{\mu\nu} - \mathfrak{F}^{\alpha\nu\lambda} \Gamma_{\alpha \cdot \lambda}^{\ast \mu} - \mathfrak{F}^{\mu\alpha\lambda} \Gamma_{\alpha \cdot \lambda}^{\ast \nu} - \frac{d}{dx^{\lambda}} \mathfrak{F}^{\mu\nu\lambda},$$

where the following abbreviations are introduced:

$$\mathfrak{F}^{\mu\nu\lambda} \stackrel{\text{def}}{=} \frac{\partial \mathfrak{L}}{\partial \Phi_{\mu|\nu\lambda}}; \quad F^{\mu\nu\lambda} = \frac{1}{\sqrt{|g|}} \mathfrak{F}^{\mu\nu\lambda},$$

and

$$\mathfrak{F}^{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \mathfrak{L}}{\partial \Phi_{\mu|\nu}}; \quad F^{\mu\nu} = \frac{1}{\sqrt{|g|}} \mathfrak{F}^{\mu\nu}.$$

Since

$$\nabla_\lambda F^{\mu\nu\lambda} = \frac{d}{dx^\lambda} F^{\mu\nu\lambda} + F^{\lambda\alpha\nu\lambda} \Gamma_{\alpha\cdot\lambda}^{\bullet\mu} + F^{\lambda\mu\alpha\lambda} \Gamma_{\alpha\cdot\lambda}^{\bullet\nu} + F^{\lambda\mu\nu\alpha} \Gamma_{\alpha\cdot\lambda}^{\bullet\lambda},$$

and owing to

$$\begin{aligned} \frac{d}{dx^\lambda} \sqrt{|g|} &= \partial_\lambda \sqrt{|g|} - (\partial_{\varrho} \sqrt{|g|}) F \Gamma_{0\cdot\lambda}^{\bullet\varrho} = \partial_\lambda \sqrt{|g|} + \frac{\partial \sqrt{|g|}}{\partial t^{\varrho}} \partial_\lambda t^{\varrho} = \\ &= \frac{1}{2} \sqrt{|g|} g^{\sigma\tau} \left\{ \partial_\lambda g_{\sigma\tau} + \frac{\partial g_{\sigma\tau}}{\partial t^{\varrho}} \partial_\lambda t^{\varrho} \right\} = \sqrt{|g|} \Gamma_{\alpha\cdot\lambda}^{\bullet\alpha}, \end{aligned}$$

we have

$$\sqrt{|g|} \nabla_\lambda F^{\mu\nu\lambda} = \frac{d}{dx^\lambda} \mathfrak{F}^{\mu\nu\lambda} + \mathfrak{F}^{\alpha\nu\lambda} \Gamma_{\alpha\cdot\lambda}^{\bullet\mu} + \mathfrak{F}^{\mu\alpha\lambda} \Gamma_{\alpha\cdot\lambda}^{\bullet\nu},$$

therefore we obtain

$$\theta^{\mu\nu} = \mathfrak{F}^{\mu\nu} - \sqrt{|g|} \nabla_\lambda F^{\mu\nu\lambda}.$$

Furthermore, introducing the notation

$$\mathfrak{F}^\mu \stackrel{\text{def}}{=} \frac{\partial \mathfrak{L}}{\partial \Phi_\mu}; \quad F^\mu = \frac{1}{\sqrt{|g|}} \frac{\partial \mathfrak{L}}{\partial \Phi_\mu},$$

we can put our equation (14.4) in the form

$$\mathfrak{F}^\mu - [\mathfrak{F}^{\varrho\nu} - \sqrt{|g|} \nabla_\lambda F^{\varrho\nu\lambda}] \Gamma_{\varrho\cdot\nu}^{\bullet\mu} - \frac{d}{dx^\nu} [\mathfrak{F}^{\mu\nu} - \sqrt{|g|} \nabla_\lambda F^{\mu\nu\lambda}] = 0,$$

and

$$\mathfrak{F}^\mu - \mathfrak{A}^{\varrho\nu} \Gamma_{\varrho\cdot\nu}^{\bullet\mu} - \frac{d}{dx^\nu} \mathfrak{A}^{\mu\nu},$$

respectively, where

$$\mathfrak{A}^{\mu\nu} \stackrel{\text{def}}{=} \mathfrak{F}^{\mu\nu} - \sqrt{|g|} \nabla_\lambda F^{\mu\nu\lambda}.$$

Since

$$\nabla A^{\mu\nu} = \frac{d}{dx^\nu} A^{\mu\nu} + A^{\varrho\nu} \Gamma_{\varrho\cdot\nu}^{\bullet\mu} + A^{\mu\varrho} \Gamma_{\varrho\cdot\nu}^{\bullet\nu},$$

just as above, we have

$$\sqrt{|g|} \nabla_\nu A^{\mu\nu} = \frac{d}{dx^\nu} \mathfrak{A}^{\mu\nu} + \mathfrak{A}^{\varrho\nu} \Gamma_{\varrho\cdot\nu}^{\bullet\mu},$$

and finally

$$F^{\mu} - \nabla_{\nu} \{ F^{\mu\nu} - \nabla_{\lambda} F^{\mu\nu\lambda} \} = 0,$$

or explicitly

$$\frac{1}{\sqrt{|g|}} \frac{\partial \mathcal{L}}{\partial \Phi_{\mu}} - \nabla_{\nu} \left\{ \frac{1}{\sqrt{|g|}} \frac{\partial \mathcal{L}}{\partial \Phi_{\mu|\nu}} - \nabla_{\lambda} \left(\frac{1}{\sqrt{|g|}} \frac{\partial \mathcal{L}}{\partial \Phi_{\mu|\nu\lambda}} \right) \right\} = 0.$$

Finally, taking into account, that the determinant g of the metrical ground tensor $g_{\mu\nu}$ does not depend on Φ_{μ} and its derivatives, we have

$$(14.5) \quad \frac{\partial L}{\partial \Phi_{\mu}} - \nabla_{\nu} \left\{ \frac{\partial L}{\partial \Phi_{\mu|\nu}} - \nabla_{\lambda} \frac{\partial L}{\partial \Phi_{\mu|\nu\lambda}} \right\} = 0.$$

This equation gives for the field of direction $v^{\mu} = v^{\mu}(x)$ the explicit covariant form of the field equation of our vectorial field.

In the case when in our space the absolute parallelism of the line-elements does not exist basing on the Vargaian methods of construction of the osculating Riemannian space another version of this theory can be elaborated. However, we shall not deal with this generalization [9] because the supposition of the existence of the absolute parallelism of the line-elements seems to be realizable in the practically interesting cases too.

Finally, by introducing the inhomogeneous directional co-ordinates—as was performed in Sect. 10—the field equations can be put in another form, however, in the general case this will not be elaborated in the following.

In Riemannian space equation (14.5) has the form

$$(14.6) \quad \frac{\partial L}{\partial \Phi_{\mu}} - \bar{\nabla}_{\nu} \left\{ \frac{\partial L}{\partial \Phi_{\mu|\nu}} - \bar{\nabla}_{\lambda} \frac{\partial L}{\partial \Phi_{\mu|\nu\lambda}} \right\} = 0.$$

where $\bar{\nabla}_{\nu}$ is the differential operator of the covariant derivative in the Riemannian space.

In the pseudo-Euclidian space, where the components of the metrical ground tensor $g_{\mu\nu}^{(0)}$ are constants, we have

$$(14.7) \quad \frac{\partial L}{\partial \Phi_{\mu}} - \partial_{\nu} \frac{\partial L}{\partial \Phi_{\mu,\nu}} + \partial_{\nu} \partial_{\lambda} \frac{\partial L}{\partial \Phi_{\mu,\nu\lambda}} = 0.$$

This equation is identical with the field equation of Podolsky's generalized theory of electromagnetic field for an arbitrary Lagrangian L [26].

15. - The differential laws of conservation.

As is well known, basing on the infinitesimal transformation of co-ordinates some identities can be deduced which from the physical point of view can be interpreted as the differential laws of conservation of the field [25]. In the following also with general vectorial fields will be dealt with.

If the variation of the Lagrangian \mathfrak{L} brought about by the change of co-ordinates is investigated we must take into account the explicit dependence of the Lagrangian on the contravariant components of the metrical fundamental tensor $g^{\mu\nu}$ and their derivatives too. The derivatives of $g^{\mu\nu}$ can be found in the parameters of connections of the space and in their derivatives. But these derivatives are the partial derivatives of the $g^{\mu\nu}$ owing to which our Lagrangian density has the form

$$(15.1) \quad \mathfrak{L} = \mathfrak{L}[g^{\mu\nu}; g^{\mu\nu}_{;\alpha}; g^{\mu\nu}_{;\alpha\beta}; g^{\mu\nu}_{;\alpha(\lambda)}; g^{\mu\nu}_{;\alpha\beta(\lambda)}; \Phi_{\mu}; \Phi_{\mu|\nu}; \Phi_{\mu|\nu\lambda}],$$

with

$$g^{\mu\nu}_{;\alpha} \stackrel{\text{def}}{=} \partial_{\alpha} g^{\mu\nu} \quad \text{and} \quad g^{\mu\nu}_{;\alpha(\lambda)} \stackrel{\text{def}}{=} \partial_{(\lambda} g^{\mu\nu}_{;\alpha)}.$$

If in our space of line-elements the absolute parallelism of line-elements exists a field of direction

$$(15.2) \quad v^{\mu} = v^{\mu}(x)$$

satisfying our equation (9.7), the integral of action defined for this field of directions is

$$I = \int_{\Omega} \mathfrak{L}(x, v(x)) \, d^4x.$$

Now, we pass over to the osculating Riemannian space introduced above in Sect. 13 which has the metrical fundamental tensor

$$\gamma^{\mu\nu}(x) = g^{\mu\nu}(x, v(x));$$

therefore, basing on (9.7)

$$\gamma^{\mu\nu}_{;\alpha} = \partial_{\alpha} g^{\mu\nu} + (\partial_{(\alpha} g^{\mu\nu})(\partial_{\lambda)} v^{\lambda}) = \partial_{\alpha} g^{\mu\nu} - F(\partial_{(\alpha} g^{\mu\nu}) \Gamma^{\lambda}_{\lambda \alpha)} = \frac{dg^{\mu\nu}}{dx^{\alpha}},$$

and similarly

$$\gamma^{\mu\nu}_{;\alpha\beta} = \frac{d^2 g^{\mu\nu}}{dx^{\alpha} dx^{\beta}}.$$

Furthermore, introducing the notations

$$\varphi_\mu(x) \stackrel{\text{def}}{=} \Phi_\mu(x, v(x)) ; \quad \varphi_{\mu;v} \stackrel{\text{def}}{=} \bar{\nabla}_v \Phi_\mu ; \quad \varphi_{\mu;v\lambda} = \bar{\nabla}_\lambda \varphi_{\mu;v} ; \quad \mathcal{L}(x) \stackrel{\text{def}}{=} \mathcal{L}(x, v(x)) ,$$

the integral of action in the osculate Riemannian space is

$$I = \int_{\Omega} \mathcal{L}(x) d^4x .$$

The total variation of I subjected to the restrictions that the ξ'' -s of the infinitesimal transformation (13.3) at the boundary of the domain of integration Ω vanish is given by

$$\delta I = \int_{\Omega} \delta^* \mathcal{L} d^4x ,$$

where

$$\delta^* \mathcal{L} = [\mathfrak{L}]_{\mu\nu} \delta^* \gamma^{\mu\nu} + \sqrt{|\gamma|} \left\{ \frac{\partial \bar{\mathcal{L}}}{\partial \varphi_\mu} - \bar{\nabla}_v \frac{\partial \bar{\mathcal{L}}}{\partial \varphi_{\mu;v}} + \bar{\nabla}_v \bar{\nabla}_\lambda \frac{\partial \bar{\mathcal{L}}}{\partial \varphi_{\mu;v\lambda}} \right\} \delta^* \varphi_\mu ,$$

being $\bar{\mathcal{L}}$ the Lagrangian in the osculating Riemannian space and

$$[\mathcal{L}]_{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \gamma^{\mu\nu}} - \partial_\alpha \left\{ \frac{\partial \mathcal{L}}{\partial \gamma^{\mu\nu}{}_{,\alpha}} - \partial_\beta \frac{\partial \mathcal{L}}{\partial \gamma^{\mu\nu}{}_{,\alpha\beta}} \right\} ,$$

the *Lagrangian derivative* of \mathcal{L} .

Assuming that the potentials Φ_μ and the corresponding potentials φ_μ in the osculating Riemannian space, respectively, fulfil the equation of field

$$\frac{\partial \bar{\mathcal{L}}}{\partial \varphi_\mu} - \bar{\nabla}_v \frac{\partial \bar{\mathcal{L}}}{\partial \varphi_{\mu;v}} + \bar{\nabla}_v \bar{\nabla}_\lambda \frac{\partial \bar{\mathcal{L}}}{\partial \varphi_{\mu;v\lambda}} = 0 ,$$

the variation of our integral of action is reduced to

$$\delta I = \int_{\Omega} [\mathcal{L}]_{\mu\nu} \delta^* \gamma^{\mu\nu} d^4x .$$

However, I is an invariant of the changes of co-ordinates, therefore

$$\delta I \equiv 0$$

for the infinitesimal transformation of co-ordinates too. This means that

$$(15.3) \quad \int_{\Omega} [\mathcal{L}]_{\mu\nu} \delta^* \gamma^{\mu\nu} d^4x \equiv 0 ,$$

for an arbitrary choice of the integration domain. But $\delta^* \gamma^{\mu\nu}$ is symmetrical in its indices μ and ν , therefore, the antisymmetrical part of the Lagrangian derivatives of \mathcal{L} does not come into consideration. Taking into account as matters stand we shall introduce the symmetric tensor density

$$(15.4) \quad \bar{\mathfrak{T}}_{\mu\nu} \stackrel{\text{def}}{=} -2 \text{Sym} ([\mathcal{L}]_{\mu\nu}) = -\{[\mathcal{L}]_{\mu\nu} + [\mathcal{L}]_{\nu\mu}\},$$

we have

$$\int_{\Omega} \bar{\mathfrak{T}}_{\mu\nu} \delta^* \gamma^{\mu\nu} d^4x \equiv 0,$$

and finally—using our equation (13.8) for $\delta^* \gamma^{\mu\nu}$ —this integral can be written in the form

$$\varepsilon \int_{\Omega} \left\{ \bar{\mathfrak{T}}_{\mu\nu} \left(\frac{d}{dx^\lambda} \gamma^{\mu\nu} \right) \xi^\lambda - 2 \bar{\mathfrak{T}}_{\mu\lambda} \left(\frac{d}{dx^\lambda} \xi^\mu \right) \right\} d^4x \equiv 0.$$

Subjected to the restriction that ξ^μ vanishes at the boundary of the domain of integration by partial integration it is obtained that

$$\varepsilon \int_{\Omega} \left\{ \bar{\mathfrak{T}}_{\mu\nu} \left(\frac{d}{dx^\lambda} \gamma^{\mu\nu} \right) + 2 \left(\frac{d}{dx^\lambda} \bar{\mathfrak{T}}_{\lambda}{}^\mu \right) \right\} \xi^\lambda d^4x = 0.$$

However, this is an identity for arbitrary ξ^μ -s and for arbitrary choice of the integration region, therefore, basing on this consideration we have

$$(15.5) \quad \frac{d}{dx^\mu} \bar{\mathfrak{T}}_{\lambda}{}^\mu - \frac{1}{2} \bar{\mathfrak{T}}^{\mu\nu} \frac{d}{dx^\lambda} \gamma_{\mu\nu} = 0,$$

and

$$(15.6) \quad \frac{d}{dx^\mu} \bar{\mathfrak{T}}_{\lambda}{}^\mu - \bar{\Gamma}_{\lambda}{}^e{}_\sigma \bar{\mathfrak{T}}_e{}^\sigma = 0,$$

respectively.

This identities are deduced in the osculating Riemannian space. Now, returning from the osculating Riemannian space to our original space of line-elements we obtain, owing to (13.2) that

$$(15.7) \quad \partial_\mu \mathfrak{T}_{\lambda}{}^\mu - F(\partial_{(q)} \mathfrak{T}_{\lambda}{}^\mu) \Gamma_{0\mu}^*{}^e - \Gamma_{\lambda\sigma}^*{}^e \mathfrak{T}_e{}^\sigma = 0$$

for the field of directions satisfying our equation (9.7).

In Riemannian space, where $g_{\mu\nu} = g_{\mu\nu}(x)$ equation (15.7) has the form

$$(15.8) \quad \partial_\mu \mathfrak{T}_{\lambda}{}^\mu - \bar{\Gamma}_{\lambda\sigma}^e \mathfrak{T}_e{}^\sigma = 0$$

and in pseudo-Euclidian space, where $\overset{(0)}{g}_{\mu\nu}$ does not depend on the co-ordinates we have

$$(15.9) \quad \partial_{\mu} T_{\lambda}^{\mu} = 0.$$

Our equations (15.7)–(15.9) are the required identities which will determine the laws of conservation for the physical field.

16. – The metrical tensor of energy and impulse.

The metrical tensor of energy and impulse of the field was originally defined by D. HILBERT [6] in the Riemannian space as the coefficients of the $\delta^* \bar{g}_{\mu\nu}$ -s in the integral

$$(16.1) \quad \delta I = 2 \int_{\Omega} [\bar{\mathfrak{L}}]^{\mu\nu} \delta^* \bar{g}_{\mu\nu} d^4x,$$

or explicitly

$$(16.2) \quad \bar{T}^{\mu\nu} = \frac{2}{\sqrt{|\bar{g}|}} [\bar{\mathfrak{L}}]^{\mu\nu}.$$

Now, owing to the identity

$$\delta^* \bar{g}_{\mu\nu} = -\bar{g}_{\alpha\mu} \bar{g}_{\beta\nu} \delta^* \bar{g}^{\alpha\beta}$$

(16.1) becomes

$$(16.3) \quad \delta I = -2 \int_{\Omega} [\bar{\mathfrak{L}}]_{\mu\nu} \delta^* \bar{g}^{\mu\nu} d^4x,$$

and similarly the covariant components of the tensor of energy and impulse can be defined as

$$(16.4) \quad T_{\mu\nu} = -\frac{2}{\sqrt{|\bar{g}|}} [\bar{\mathfrak{L}}]_{\mu\nu}.$$

It can be proved easily that $\bar{T}^{\mu\nu}$ and $\bar{T}_{\mu\nu}$, respectively, are the contravariant, and the covariant components respectively, of the *same* tensor.

These considerations were valid in the Riemannian space. To define the metrical tensor of energy and impulse in the space of line-elements too—assuming that in the space of line-elements the absolute parallelism of the line-elements exists we shall suppose that there is given a field of direction $v^{\mu} = v^{\mu}(x)$ fulfilling the equations (9.7)—we introduce the metrical tensor of energy-

impulse (owing to (15.4) by definition)

$$(16.5) \quad T_{\mu\nu} \stackrel{\text{def}}{=} - \frac{2}{\sqrt{|g|}} \left\{ \frac{\partial \mathfrak{L}}{\partial g^{\mu\nu}} - \frac{d}{dx^\alpha} \left[\frac{\partial \mathfrak{L}}{\partial g^{\mu\nu}{}_{,\alpha}} - \frac{d}{dx^\beta} \frac{\partial \mathfrak{L}}{\partial g^{\mu\nu}{}_{,\alpha\beta}} \right] \right\}.$$

Taking into account that

$$\mathfrak{L} = \sqrt{|g|} \cdot L,$$

we have

$$(16.6) \quad T_{\mu\nu} = g_{\mu\nu} L - \left\{ \frac{\partial L}{\partial g^{\mu\nu}} - \frac{d}{dx^\alpha} \left[\frac{\partial L}{\partial g^{\mu\nu}{}_{,\alpha}} - \Gamma_{\beta \cdot \sigma}^{\alpha \sigma} \frac{\partial L}{\partial g^{\mu\nu}{}_{,\alpha\beta}} - \frac{d}{dx^\beta} \frac{\partial L}{\partial g^{\mu\nu}{}_{,\alpha\beta}} \right] - \right. \\ \left. - \Gamma_{\alpha \cdot \tau}^{\alpha \tau} \left[\frac{\partial L}{\partial g^{\mu\nu}{}_{,\alpha}} - \Gamma_{\beta \cdot \sigma}^{\alpha \sigma} \frac{\partial L}{\partial g^{\mu\nu}{}_{,\alpha\beta}} - \frac{d}{dx^\beta} \frac{\partial L}{\partial g^{\mu\nu}{}_{,\alpha\beta}} \right] \right\},$$

defined for the field of directions $v^\mu = v^\mu(x)$ where the differential operator d/dx^α is the Berwald's derivative introduced above.

In Riemannian space the tensor of energy and impulse became

$$(16.7) \quad \bar{T}_{\mu\nu} = \bar{g}_{\mu\nu} \bar{L} - \left\{ \frac{\partial \bar{L}}{\partial \bar{g}^{\mu\nu}} - \partial_\alpha \left[\frac{\partial \bar{L}}{\partial \bar{g}^{\mu\nu}{}_{,\alpha}} - \bar{\Gamma}_{\beta \cdot \sigma}^{\alpha \sigma} \frac{\partial \bar{L}}{\partial \bar{g}^{\mu\nu}{}_{,\alpha\beta}} - \partial_\beta \frac{\partial \bar{L}}{\partial \bar{g}^{\mu\nu}{}_{,\alpha\beta}} \right] - \right. \\ \left. - \bar{\Gamma}_{\alpha \cdot \tau}^{\alpha \tau} \left[\frac{\partial \bar{L}}{\partial \bar{g}^{\mu\nu}{}_{,\alpha}} - \bar{\Gamma}_{\beta \cdot \sigma}^{\alpha \sigma} \frac{\partial \bar{L}}{\partial \bar{g}^{\mu\nu}{}_{,\alpha\beta}} - \partial_\beta \frac{\partial \bar{L}}{\partial \bar{g}^{\mu\nu}{}_{,\alpha\beta}} \right] \right\},$$

and finally in the pseudo-Euclidian space

$$(16.8) \quad T_{\mu\nu}^{(0)} = g_{\mu\nu}^{(0)} [L] g^{\mu\nu} = g^{\mu\nu (0)} - \left\{ \frac{\partial L}{\partial g^{\mu\nu}} - \partial_\alpha \frac{\partial L}{\partial g^{\mu\nu}{}_{,\alpha}} + \partial_\alpha \partial_\beta \frac{\partial L}{\partial g^{\mu\nu}{}_{,\alpha\beta}} \right\} g^{\mu\nu (0)} = g^{\mu\nu (0)},$$

where L is the Lagrangian of the field in the corresponding Riemannian space.

The metrical tensor of energy and impulse of the field (16.5) was defined on the basis of (15.4) passing from the osculating Riemannian space to the space of line-elements. Therefore, in the different spaces the tensors (16.6)–(16.8) satisfy the identities (15.8)–(15.10), respectively, representing also the required laws of conservation of energy and impulse.

The above considerations were based on the assumption that in our space of line-elements the absolute parallelism of line-elements exists. The tensor of energy and momentum was defined in this case and it was shown that this tensor fulfils the laws of conservation. If the absolute parallelism of line-elements does not exist the tensor of energy and momentum (16.6) can be defined by the definition (16.6), however, this $T_{\mu\nu}$ does not fulfil the identity (15.8) and, therefore, it seems not possible to give $T_{\mu\nu}$ some physical meaning.

* * *

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APPENDIX (*)

Space-time structure and the violation of parity conservation.

The violation of parity conservation predicted by LEE and YANG [39] — it has been illustrated by the cobalt experiment of WU *et al.* [40] — was recently investigated from the point of view of relativistic invariance by WIGNER [41].

As is well-known, the consequence of this and other experiments may be summarized briefly saying that the symmetry of the real world is smaller than it had been thought. Namely, the whole experimental arrangement in the case of ^{60}Co has at the beginning of the experiment a symmetry plane, which should remain valid throughout the further fate of the system. Nevertheless, the intensity of the β radiation is larger on one side of this plane than on the other one [42]. If it is, however, true that a symmetry plane remains a symmetry plane, the initial state of the cobalt experiment could not have contained a symmetry plane. Therefore, the radical solution of this paradoxical situation was suggested by WIGNER who stated that the polar and axial transformation character of the electric and magnetic vector of the electromagnetic field has to be changed, with the immediate consequence that the charge density would become a pseudo-scalar rather than a scalar quantity as in the current theory. By this means the problem concerning the cobalt experiment would be solved, and the CP invariance predicted by LANDAU [43] would also be explained, however, this suggestion should entail further consequences that are at present difficult to foresee and it is inconceivable that such a radical change in the case of the theory of the electromagnetic field should not have been suggested by any previous experimental effect.

Owing to our above considerations no less radical interpretation of the cobalt experiment and of other experiments which prove the violation of parity conservation, respectively, can be suggested in the expectation that it does not result any alarming consequence of the same kind. Let us namely suppose that *by effects (especially in the case of weak interactions) violating the parity conservation the isotropy of the space part (or more generally that of a space-like surface) of the space-time world would be destroyed.*

Due to this supposition the anisotropy of the space would be determined, at first insight, by the experimental test quite phenomenologically. Let us suppose that the surface corresponding with the probability of β -decay per unit solid angle and per unit electron energy should be identical with the indicatrix of the anisotropic space. This means in terms of the suggested theory that the equation of this indicatrix in its parametric form is determined explicitly by

$$\begin{aligned}\alpha^1 &= S(E)1 \{ -a(E) \cos \vartheta \} \sin \vartheta \cos \varphi, & \alpha^2 &= S(E) \{ 1 - a(E) \cos \vartheta \} \sin \vartheta \sin \varphi, \\ \alpha^3 &= S(E) \{ 1 - a(E) \cos \vartheta \} \cos \vartheta,\end{aligned}$$

(*) Added in proof on 10th August 1958.

(where E is the electron energy and ϑ is the angle between the electron momentum and the spin direction of the decaying nucleus [42]), and the components of the metrical fundamental tensor of the space is than given by

$$g_{ik}(v^1, v^2, v^3) = \delta_{ik} S(E) \{1 - a(E) v^3 [(v^1)^2 + (v^2)^2 + (v^3)^2]^{-\frac{1}{2}}\}.$$

Such a space (*) is the special case of the general line-element space elaborated above.

Obviously, because of only geometrical facts, it is clear that the reflexion symmetry in an anisotropic space is destroyed, however, in the case of CP transformation not only the sign of v^3 , but also that of $a(E)$ is changed in the formula of the metrical fundamental tensor, which may be a natural geometrical interpretation of Landau's theory.

Since, it is well known that in the case of the longitudinal theory of neutrino the asymmetry parameter $a(E)$ has the order of magnitude w/c (where w is the mean velocity of the emitted electrons and c is that of light), this means that the anisotropy of the space tends to isotropy in the limiting case when the velocity (*i.e.* the kinetic energy) of the emitted electrons tends to zero.

The suggested theory based on the mentioned phenomenological determination of the anisotropy of the space has, however, two alarming objectionable points: (i) the asymmetry parameter, if the suggested *a priori* anisotropy of the space is true at all, should be rather deduced from a law of nature than on the basis of the above phenomenological reasoning; (ii) it must be explained that different asymmetry parameters follow from different decay processes.

These objections have to be eliminated more accurately in the course of future investigations. Some months ago, when the belief was generally accepted that the violation of parity conservation is only typical of weak interactions, it was possible to suppose that the anisotropy of the space at the position of the decay process depends on the number of the created neutrinos (n) and for the asymmetry parameter the formula of interpolation $a(E) = f(E) \{n^{-2} + 1 + 7(n-1)^2\}^{-1}$ could have been proposed. At present, however, when the violation of parity conservation also in the case of strong interactions seems to be proved, the supposition can only be suggested that the asymmetry parameter depends also on the internal structure of the interacting fields (in the sense of Sect. 9) and on the charges as well as on the masses of the interacting particles, respectively. But, unfortunately, the present observational material does not render possible the determination of a — just heuristical — formula for the asymmetry parameter and a start of further theoretical investigations.

However, the suggested theory in terms of the unified theory of fields can be interpreted as follows: owing to Mach's axiom that *the laws of nature depend on the physical content of the universe*, one can suggest that in the case of the weakest interaction (*i.e.* of gravitational interaction) the homogeneity and in the case of weak interactions as well as in that of interactions, where also

(*) This space has some interest also from the geometrical point of view. This is, namely, an explicit example for a space, all its three curvature tensors vanish, but it is non-Euclidian since its torsion tensor is different from zero.

the internal degree of freedom of the interacting fields became dominant, the isotropy of the space would be destroyed. Therefore, let us conclude that *the deviation of the structure of the space from its Euclidian character depends inversely proportional to the coupling of the interactions*. By this means one can understand that the isotropy of the space (i.e. in Wigner's terms: the law of symmetry) becomes dominant only in the case of electromagnetic and especially in that of mesonic interaction when the inhomogeneity and anisotropy of the space is more and more overcompensated by the strength of the external interaction submitted by the physical fields of second kind.

Finally, Wigner's suggestion — and, as he has pointed out, the spirit of the ideas of Lee and Yang — that all symmetry properties would be only approximate, should be replaced by the geometrized supposition that *isotropy and homogeneity of the space, which was accepted a priori without any criticism in the current field theories, would be only an approximate one and by the decrease of the strength of interactions the anisotropy and inhomogeneity of the space-time world would become more and more dominant*.

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Studies of the Spherical Harmonics Method in Neutron Transport Theory.

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CONTENTS. — I. The relation between P_L - and Gauss quadrature solutions of the Milne problem. — Introduction. — 1. The P_L -approximation. — 2. Scattering involving higher atomic angular momenta. — 3. The characteristic equation. — 4. The solution on the vacuum and the medium side. — 5. A short comparison with the Gauss quadrature method. — 6. Completion of the solution. — 7. Investigation of $f^{\text{II}}(0, \mu)$. — 8. Density and current. — 9. Special cases. — APPENDIX I. Calculation of the sum over l in equation (15). — APPENDIX II. Calculation of the coefficients α_k . — APPENDIX III. Summation of the C-W part $f^{\text{I}}(0, \mu)$. — APPENDIX IV. Calculation of the current at the boundary. — II. Behavior of the P_L - solution of the Milne problem with anisotropic scattering for $L \rightarrow \infty$. (Algebraic part). — 1. The error-source in the Boltzmann equation caused by spherical harmonics solutions. — 2. The functions, which tend for positive and negative μ in the limit $L \rightarrow \infty$ to the exact directed flux.

I. — The Relation between P_L - and Gauss quadrature solutions of the Milne problem.

Introduction.

The fact that the Gauss quadrature (denoted here as the Chandrasekhar-Wick [C-W]) method and the spherical harmonics method of dealing with the *monoenergetic* transport equation are closely related, has been often pointed

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out in the literature. However, contrary to some statements which have been made, the two are not identical—basically the C-W method, as applied to the Milne problem, is a non-analytic approximation, whereas all functions used in the spherical harmonics method are continuous. It is the purpose of this paper to examine in detail the relation between the two methods. The Milne problem with p -wave scattering will be used as a convenient example for the analysis. Incidental to these considerations will be derived completely general spherical harmonics solutions to the Milne problem, and a comparison with the Gauss quadrature solutions will be given. In a following paper it will be shown how these approximate results, in the limit, give the rigorous results of the Wiener-Hopf method of solution of the Milne problem.

Previous general solutions of the Milne problem by application of the spherical harmonics method have been given by J. C. MARK ⁽¹⁾, by B. CARLSON ⁽²⁾, and by M. C. WANG and E. GUTH ⁽³⁾. The considerations which follow are more general than those of previous authors, first, in that the scattering is assumed anisotropic, and second, in that the relation with the Gauss quadrature method is shown more clearly.

1. — The P_L -approximation.

Suppose a medium which occupies the region $z \leq 0$ has s - and p -wave scattering of neutrons and has total macroscopic cross-section $\Sigma = \Sigma_a + \Sigma_s$, where Σ_a and Σ_s are the cross-sections for absorption and scattering. The anisotropy of the scattering is covered by a transport cross-section Σ_t . The x, y -plane is a plane boundary between the medium and the vacuum $z \geq 0$.

Transport theory yields the integro-differential equation

$$\mu \frac{\partial f(z, \mu)}{\partial z} + \Sigma f(z, \mu) = \frac{1}{2} \Sigma_s \int_{-1}^{+1} f(z, \mu') d\mu' + \frac{3}{2} (\Sigma - \Sigma_t) \mu \int_{-1}^{+1} f(z, \mu') \mu' d\mu',$$

where $\mu = \cos \vartheta$ and ϑ is the angle between the positive z -axis and the direction of the directed flux $f(z, \mu)$. At $z = -\infty$ there is an infinite source of neutrons. It is convenient, to introduce dimensionless quantities $\zeta = \Sigma z$ for the co-ordinate, $\gamma_a = \Sigma_a/\Sigma$, $\gamma_t = \Sigma_t/\Sigma$ and later $c = 1 - \gamma_a = \Sigma_s/\Sigma$, $a = 3\gamma_a(1 - \gamma_t)$. The Boltzmann equation takes then the form

$$(1) \quad \mu \frac{\partial f(\zeta, \mu)}{\partial \zeta} + f(\zeta, \mu) = \frac{1}{2} (1 - \gamma_a) \int_{-1}^{+1} f(\zeta, \mu') d\mu' + \frac{3}{2} (1 - \gamma_t) \mu \int_{-1}^{+1} f(\zeta, \mu') \mu' d\mu'.$$

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⁽³⁾ M. C. WANG and E. GUTH: *Phys. Rev.*, **84**, 1092 (1951).

Applying now the spherical harmonics method, by assuming first that

$$(2) \quad f(\zeta, \mu) = \sum_{l=0}^{\infty} f_{l0}(\zeta) P_l(\mu),$$

one finds for the partial fluxes ($4\pi f_{00}(\zeta)$ = density, $(4\pi/3)f_{10}(\zeta)$ = current in z -direction, $f_{20}(\zeta)$, $f_{30}(\zeta)$, ...) the following *infinite* system of differential equations ($' = \partial/\partial\zeta$):

$$(3) \quad \begin{cases} l=0 & \frac{1}{3} f'_{10} & + \gamma_a f_{00} = 0, \\ l=1 & \frac{2}{5} f'_{20} + f'_{00} + \gamma_i f_{10} = 0, \\ l=2 & \frac{3}{7} f'_{30} + \frac{2}{3} f'_{10} + f_{20} = 0, \\ l=3 & \frac{4}{9} f'_{40} + \frac{3}{5} f'_{20} + f_{30} = 0, \\ \vdots & \\ l & \frac{l+1}{2l+3} f'_{l+1,0} + \frac{l}{2l-1} f'_{l-1,0} + f_{l0} = 0. \\ \vdots & \end{cases}$$

The general equation l holds for $l \geq 2$; the first two equations for $l=0, 1$, however, are changed or «disturbed» if $\gamma_a \neq 1$ and $\gamma_i \neq 1$. One may describe the vacuum as far as its effect on the medium is concerned, as a perfect absorber without scattering, *i.e.* $\gamma_a = 1$, $\gamma_i = 1$ in the vacuum. In this case the general equation holds for all $l \geq 0$, because $f_{-1,0}(\zeta) = 0$; its solution in this case will be simpler than in the general case.

In the L -approximation of the spherical harmonics method one cuts the infinite system of differential equations after the L th-equation and demands that only the spherical harmonics up to and including P_L shall be used in the series (2) as an approximation. This implies that $f_{L+1,0}(\zeta) = 0$, $f_{L+2,0}(\zeta) = 0$, $f_{L+3,0}(\zeta) = 0$, ... The first of these equations, $f_{L+1,0}(\zeta) = 0$, turns out to be equivalent to the characteristic equation, which determines the various modes of partial solutions of the finite system left after the truncation of the infinite system of differential equations. To put the remaining $f_{L+2,0}(\zeta)$, $f_{L+3,0}(\zeta)$, ... = 0 is a process, which is, properly taken, *inconsistent with the infinite system of differential equations*, because these later harmonics are related to the non-vanishing $f_{L,0}(\zeta)$ by a recursion formula which keeps them different from zero until $l \rightarrow \infty$. Indeed, putting $f_{L+1,0} = 0$ in the equ. $l = L+1$ of the system (3), one obtains

$$[(L+2)/(2L+5)] f'_{L+2,0} + [(L+1)/(2L+1)] f'_{L0} = 0.$$

$f_{L+2,0} = 0$ does not satisfy this equation, because f'_{L0} is in P_L -approximation certainly different from zero.

Only after addition of a term $[(L+1)/(2L+1)] f'_{L0}$ on the right hand side of this equation it will be allowed to put $f_{L+2,0} = 0$. Then a glance at the further equations of the

system (3) shows the legality of putting $f_{L+3,0}=f_{L+4,0}=f_{L+5,0}=...=0$ also. Therefore the P_L -approximation fulfils a Boltzmann-equation (1) with an additional L -dependent source-term

$$[(L+1)/(2L+1)](df_{L0}/d\xi)P_{L+1}(\mu)$$

on its right hand side. This « error-source-term » leads to non-uniform convergence of the spherical harmonic approximations to the exact directed flux for $L \rightarrow \infty$, e.g. in the case of the Milne problem investigated further in part II, 1.

The supposition

$$(4) \quad f_{l0}(\zeta) = (2l+1)g_l(\lambda) \exp[-\zeta/\lambda] \quad (l \geq 2)$$

in the l -th equation of (3) yields a recursion formula for the g_l 's

$$(l+1)g_{l+1}(\lambda) + lg_{l-1}(\lambda) - (2l+1)\lambda g_l(\lambda) = 0,$$

which is identical with the recursion formula of the Legendre functions of the 1st and 2nd kind. Therefore, putting

$$(5) \quad f_{l0}(\zeta) = (2l+1)[A(\lambda)P_l(\lambda) + B(\lambda)Q_l(\lambda)] \exp[-\zeta/\lambda],$$

with 2 arbitrary functions $A(\lambda)$ and $B(\lambda)$, independent of l , one is able to fulfil the set of differential equations (3) for $2 \leq l \leq \infty$. The parameter λ is arbitrary until now and the solution f_{l0} above is a partial solution only. From the Legendre function of second kind (⁴)

$$Q_l(\lambda) = P_l(\lambda)Q_0(\lambda) - W_{l-1}(\lambda) \quad \text{with} \quad Q_0(\lambda) = \frac{1}{2} \log \frac{1+\lambda}{1-\lambda} \quad (|\lambda| < 1),$$

one must use only the non-singular part $W_{l-1}(\lambda)$, in order to avoid solutions which are singular at $\lambda = \pm 1$ and to get an algebraic characteristic equation. W_{l-1} satisfies the same recursion formula as P_l with the index on W lowered by 1; this is evident since Q_l and P_l satisfy the recursion relation and Q_0 , being independent of l , does not disturb the recursion. Hence, one adopts simply the pure polynomial

$$(6) \quad f_{l0}(\zeta) = (2l+1)[A(\lambda)P_l(\lambda) + B(\lambda)W_{l-1}(\lambda)] \exp[-\zeta/\lambda]$$

as a partial solution for $l \geq 2$. The functions $A(\lambda)$ and $B(\lambda)$ are to be determined by a calculation of $f_{20}(\zeta)$ and $f_{30}(\zeta)$ as a consequence of the first two « disturbed » equations for $l=0$ and $l=1$. Supposing

$$f_{00}(\zeta) = \exp[-\zeta/\lambda]$$

(⁴) E. JAHNKE and F. EMDE: *Tables of Functions*, 4th Ed., Ch. VII.

one obtains from equations (3) for

$$\begin{aligned} l = 0: & \quad f_{10}(\zeta) = 3\gamma_a \lambda \exp[-\zeta/\lambda], \\ l = 1: & \quad f_{20}(\zeta) = 5[-\tfrac{1}{2} + \tfrac{3}{2}\gamma_a \gamma_t \lambda^2] \exp[-\zeta/\lambda], \\ l = 2: & \quad f_{30}(\zeta) = 7[-(\tfrac{5}{6} + \tfrac{2}{3}\gamma_a)\lambda + \tfrac{5}{2}\gamma_a \gamma_t \lambda^3] \exp[-\zeta/\lambda]. \end{aligned}$$

A comparison of these $f_{20}(\zeta)$ and $f_{30}(\zeta)$ with those of equation (6) yields

$$(7) \quad \begin{cases} A(\lambda) = 1 + 3\gamma_a(1 - \gamma_t)\lambda^2 & = 1 + a\lambda^2, \\ B(\lambda) = -\lambda[1 - \gamma_a + 3\gamma_a(1 - \gamma_t)\lambda^2] & = -\lambda(c + a\lambda^2). \end{cases}$$

If we characterize the vacuum by $\gamma_a = 1$ and $\gamma_t = 1$, $B_{\text{vac}}(\lambda)$ vanishes and we have only Legendre polynomials of 1st kind with the coefficients $A_{\text{vac}}(\lambda) = 1$. Hence, the vacuum is described by the *simplest* solutions. The usual Milne problem has vacuum on one side; but one may see that the solutions above are adapted to two or even more media with arbitrary different γ_a , γ_t 's.

This method of connecting the solutions $f_{l0}(\zeta)$ with a supposed partial solution $f_{00}(\zeta)$ was already used by B. CARLSON in the case of isotropic scattering.

2. - Scattering involving higher atomic angular momenta.

Let us interrupt briefly the course of further calculations to show that this method of determination of $A(\lambda)$ and $B(\lambda)$ may be extended to d , f , g , ...-wave scattering. Then we shall have more «disturbed» equations in the system (3). We deal with d -wave scattering by putting a factor γ_d before f_{20} in equation $l = 2$, with f -wave scattering by putting a factor γ_f before f_{30} in equation $l = 3$, and so on.

If \bar{l} is the first «undisturbed» equation with a factor 1 before f_{l0} namely $l = \bar{l}$:

$$\frac{\bar{l} + 1}{2\bar{l} + 3} f'_{\bar{l}+1,0} + \frac{\bar{l}}{2\bar{l} - 1} f_{\bar{l}-1,0} + f_{\bar{l}0} = 0,$$

then $A(\lambda)$ and $B(\lambda)$ have to be determined from the equations

$$\frac{f_{\bar{l}0}}{(2\bar{l} + 1)f_{00}} = A(\lambda) P_{\bar{l}}(\lambda) + B(\lambda) W_{\bar{l}-1}(\lambda),$$

$$\frac{f_{\bar{l}+1}}{(2\bar{l} + 3)f_{00}} = A(\lambda) P_{\bar{l}+1}(\lambda) + B(\lambda) W_{\bar{l}}(\lambda).$$

After the application of the relation

$$P_{\bar{l}}(\lambda) W_{\bar{l}}(\lambda) - P_{\bar{l}+1}(\lambda) W_{\bar{l}-1}(\lambda) = \frac{1}{\bar{l} + 1},$$

the solution of these equations acquires the simple form

$$(8) \quad \begin{cases} A(\lambda) = \frac{\bar{l} + 1}{f_{00}} \left\{ \frac{1}{2\bar{l} + 1} f_{i0} W_{\bar{l}}(\lambda) - \frac{1}{2\bar{l} + 3} f_{i+1,0} W_{\bar{l}-1}(\lambda) \right\}, \\ B(\lambda) = -\frac{\bar{l} + 1}{f_{00}} \left\{ \frac{1}{2\bar{l} + 1} f_{i0} P_{\bar{l}+1}(\lambda) - \frac{1}{2\bar{l} + 3} f_{i+1,0} P_{\bar{l}}(\lambda) \right\}. \end{cases}$$

In a straightforward calculation one has to find f_{i0}/f_{00} and $f_{i+1,0}/f_{00}$ from the « disturbed » system $l=0$ until $l=\bar{l}-1$ and the two subsequent « undisturbed » equations $l=\bar{l}$ and $l=\bar{l}+1$. Putting the result in equations (8) one finds $A(\lambda)$ and $B(\lambda)$ in the extended case.

In the special case $\bar{l}=3$ with γ_d as a factor before f_{20} in the equation $l=2$ of the system (3) one obtains

$$(9) \quad \begin{cases} A(\lambda) = 1 + a\lambda^2 + b \left[-3\lambda^2 + 9 \left(1 - c - \frac{a}{3} \right) \lambda^4 \right], \\ B(\lambda) = -\lambda \left\{ c + a\lambda^2 + b \left[1 - 3 \left(2 - c - \frac{a}{3} \right) \lambda^2 + 9 \left(1 - c - \frac{a}{3} \right) \lambda^4 \right] \right\}, \end{cases}$$

where the additional terms with the factor

$$(10) \quad b = \frac{5}{4}(1 - \gamma_d)$$

show the influence of the d -wave scattering. One recognizes that every further « disturbed » equation increases the degree in λ of $A(\lambda)$ and $B(\lambda)$ by 2. The degree of the characteristic equation in P_L -approximation, however, always remains $L+1$ in λ , because it is independent of the constant factors $\gamma_a, \gamma_t, \gamma_d, \dots$. Consequently, some relations exist between the coefficients of all powers of λ^2 in $A(\lambda)$ and $-B(\lambda)/\lambda$ with exception of the power λ^0 . If one writes in our special case $\bar{l}=3$ $A(\lambda)$ and $B(\lambda)$ as polynomials of λ^2

$$A(\lambda) = A_1 + A_2\lambda^2 + A_3\lambda^4, \quad -\frac{B(\lambda)}{\lambda} = B_1 + B_2\lambda^2 + B_3\lambda^4,$$

one obtains in this way the relations

$$B_3 = A_3, \quad B_2 = A_2 - \frac{1}{3}A_3$$

and the characteristic equation for the $\bar{l}=3$ case (the analog of the following equation (12)) contains

$$A_1 - B_1 - \frac{1}{5}A_3 - \frac{1}{3}B_2 = \left(1 - c - \frac{a}{3} \right) \left(1 - \frac{4}{5}b \right) = \gamma_a \gamma_t \gamma_d$$

as coefficient of its highest power λ^{L+1} .

Presumably this coefficient will be simply the product of all γ 's in the general case of arbitrary $\bar{l} < L-1$ also.

3. - The characteristic equation.

We return to the case $\bar{l}=2$, i.e. to our main course. The partial solutions $f_{00}(\zeta)$, $f_{10}(\zeta)$ and $f_{i0}(\zeta)$ for $l \geq 2$, when put in equation (2), give a partial solution $f(\zeta, \mu; \lambda)$ of the infinite system (3) of differential equations, charac-

terized by an arbitrary parameter λ . A general solution could be constructed by superposition of a dense distribution of such solutions

$$f(\zeta, \mu) = \int \alpha(\lambda) f(\zeta, \mu; \lambda) d\lambda,$$

where $\alpha(\lambda)$ should be chosen so as to satisfy the boundary conditions. This process of satisfying the boundary conditions is reduced to an *algebraic* problem by cutting the infinite system (3) after a finite L in the spherical harmonics approximation. Then $L+1$ equations remain, the last one being

$$(11) \quad \frac{L+1}{2L+3} f_{L+1,0}' + \frac{L}{2L-1} f_{L-1,0}' + f_{L,0} = 0.$$

In the P_L -approximation, however, $f_{L+1,0}$ is not allowed to appear; this is only the case if one selects the λ 's of the partial solutions (5) in such a manner, that $f_{L+1,0}(\lambda_k)$ with these special λ_k 's vanishes. Therefore, the characteristic equation in L -approximation is

$$(12) \quad \left\{ \begin{array}{l} f_{L+1,0}(\lambda_k) = A(\lambda_k) P_{L+1}(\lambda_k) + B(\lambda_k) W_L(\lambda_k) = 0 \\ i.e. \\ (1 + a\lambda_k^2) P_{L+1}(\lambda_k) - \lambda_k(c + a\lambda_k^2) W_L(\lambda_k) = 0. \end{array} \right.$$

This is an algebraic equation of degree $L+1$, because the coefficients of the higher powers cancel. If L is odd, $L+1$ is even and the equation is of degree $(L+1)/2$ in λ_k^2 ; hence, one has $(L+1)/2$ pairs of roots $\lambda_k, -\lambda_k$. In case L is even, $L+1$ is odd, a root $\lambda = 0$ appears. This event is uncomfortable and the following treatment is for *odd* L only.

4. - The solutions on the vacuum and the medium side.

Having obtained the $f_{l0}(\zeta)$'s, we put them in equation (2) and obtain the solution for $f(z, \mu)$.

a) On the *vacuum side* ζ has a different meaning in comparison with the medium side; it is $\zeta = \Xi z$ with an arbitrarily chosen cross section for absorption $\Xi_a = \Xi$; $\Xi_s = 0$. After the calculation one may put $\Xi_a = \infty$ but this is not essential. The solution on this side, $z \geq 0$, is in P_L -approximation

$$(13) \quad \begin{aligned} f(z, \mu) &= \sum_{j=1}^{(L+1)/2} [\beta_j \exp[-\zeta/\mu_j] \sum_{l=0}^L (2l+1) P_l(\mu_j) P_l(\mu)] \\ &= (L+1) P_{L+1}(\mu) \sum_{j=1}^{(L+1)/2} \beta_j \frac{P_L(\mu_j)}{\mu - \mu_j} \exp[-\zeta/\mu_j], \end{aligned}$$

with arbitrary constants β_j , which are found later from the boundary conditions at $z=0$. The characteristic equation is

$$(14) \quad P_{L+1}(\mu_j) = 0,$$

from which we keep the positive roots μ_j ($j=1, 2, \dots, (L+1)/2$). One has to take positive roots only because $f(z, \mu)$ should tend to zero for $z = +\infty$; hence the solution satisfies the boundary condition of the Milne problem at $z = \infty$. From the second expression for $f(z, \mu)$ in equation (13) one sees immediately that it has zeros in the directions $\mu = -\mu_j$ for all $z \geq 0$. Hence we fulfil the condition of no backward flux into the medium only in $(L+1)/2$ discrete backward cones in the P_L -approximation. This result is valid for $z=0$ also and is essential for establishing the boundary conditions on the medium side. $f(\zeta, \mu)$ has no poles in the forward directions $\mu = +\mu_j$, because the denominator $\mu - \mu_j$ cancels against an equal factor in

$$P_{L+1}(\mu) = (2L+2)! / \{2^{L+1}[(L+1)!]^2\} \prod_{j=1}^{(L+1)/2} (\mu - \mu_j)(\mu + \mu_j).$$

In the directions of the cones $\mu = +\mu_j$ we have a finite flux for $z \geq 0$

$$\begin{aligned} f(z, +\mu_j) &= (L+1)\beta_j P'_{L+1}(\mu_j) P_L(\mu_j) \exp[-\zeta/\mu_j] = \\ &= (L+1)^2 \beta_j \frac{[P_L(\mu_j)]^2}{1-\mu_j^2} \exp[-\zeta/\mu_j]. \end{aligned}$$

b) The solution on the *medium side*, which must satisfy the boundary conditions at $z=0$ later, is in P_L -approximation, $z \leq 0$, $\zeta = \Sigma z$,

$$\begin{aligned} (15) \quad f(z, \mu) &= \sum_{k=0,1,2,\dots}^{(L-1)/2} [\alpha_k \exp[\zeta/\lambda_k] \{1 - 3\gamma_a \lambda_k \mu + \\ &+ \sum_{l=2}^L (2l+1)((1 + a\lambda_k^2)P_l(-\lambda_k) + \lambda_k(c + a\lambda_k^2)W_{l-1}(-\lambda_k))P_l(\mu)\}], \end{aligned}$$

with arbitrary constants α_k . Instead of counting the roots froms 1, 2, ..., $(L+1)/2$ in the sum (15), the largest root λ_0 is distinguished from the others $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{(L-1)/2}$, because it, being essentially the diffusion length, plays a somewhat different role in the Milne problem, which contains an infinite source at $z = -\infty$, than the other roots. To represent this fact, the sum contains one term with the coefficient α_0 and the root $\lambda_0 = -\lambda_0$. The exponential factor $\exp[\zeta/\lambda_0] = \exp[-\zeta/\lambda_0]$ in this case tends to infinity for $\zeta \rightarrow -\infty$. It is the slowest decreasing solution with negative λ 's, if one follows the positive z -direction from the source at $z = -\infty$, because λ_0 is the largest root of (12). Therefore, (15) already satisfies the boundary condition of the Milne problem at $z = -\infty$ and the extra term $k=0$ is chosen to give the asymptotic behavior of $f(\zeta, \mu)$ at $\zeta = -\infty$.

By applying Christoffel's sum formulae—see Appendix I—on the sum over l in equation (15) and by using the characteristic equation (12) one may give the equation (15) a simpler form ($z \leq 0$)

$$(16) \quad f(z, \mu) = \sum_{k=0,1,\dots}^{(L-1)/2} \alpha_k \lambda_k \exp[\zeta/\lambda_k] \left\{ -a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k + \mu} \left[1 - \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \right] \right\}.$$

One recognizes immediately from equation (15) that $f(z, \mu)$ is a *polynomial* in μ of at most the degree L in μ . The equivalent expression (16) has no poles in the directions $\mu = -\lambda_k$; the denominators $\lambda_k + \mu$ are removable again by developing $P_{L+1}(\mu) - P_{L+1}(-\lambda_k)$ in a power series of $\mu + \lambda_k$.

In the following we always denote by \sum_k a sum over $k = \bar{0}, 0, 1, \dots, (L-1)/2$ and we take an index s for summing only over $s = 1, 2, \dots, (L-1)/2$ after removing the 2 summands with $k = \bar{0}, 0$.

If we separate $f(z, \mu) = f^I(z, \mu) + f^{II}(z, \mu)$ into two parts

$$(17) \quad f^I(z, \mu) = \sum_k \alpha_k \lambda_k \exp[\zeta/\lambda_k] \left\{ -a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k + \mu} \right\}$$

$$(18) \quad f^{II}(z, \mu) = - \sum_k \alpha_k \lambda_k \exp[\zeta/\lambda_k] \frac{c + a\lambda_k^2}{\lambda_k + \mu} \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)}, \quad *$$

we have in $f^I(z, \mu)$ the result, which could be found by applying the Gauss quadrature method to the problem according to C-W for *positive* μ . $f^I(z, \mu)$ has poles, however, in the $(L+1)/2$ *negative* directions $\mu = -\lambda_k$ and must be replaced in the C-W method by another expression using the original integral equation (see part II, 2). $f^{II}(z, \mu)$ is zero for $\mu = \mu_j$; therefore the fluxes according to both methods coincide in the $(L+1)/2$ positive directions $\mu = \mu_j$ for all $z \leq 0$. The addition of $f^{II}(z, \mu)$ removes the poles of $f^I(z, \mu)$ at $\mu = -\lambda_k$ again; the spherical harmonics method unlike the Gauss quadrature method yields in $f(z, \mu)$ a continuous expression for all μ 's but at the cost of a poorer approximation compared with the Gauss quadrature method.

Considering now the boundary at $z=0$ we remark that $f(0, \mu)$ from equation (16) for $z \leq 0$ contains the $(L+3)/2$ unknown constants α_k and from equation (13) for $z \geq 0$ the $(L+1)/2$ unknown constants β_j . Therefore one of the α_k 's is still available for the normalization of the asymptotic behavior of $f(z, \mu)$ at $z = -\infty$. We impose on the first 2 coefficients the condition $\alpha_{\bar{0}} + \alpha_0 = 1$. This means that we choose the solution for the density $4\pi f_{00}(\zeta)$ in the following manner ($\zeta \leq 0$)

$$(19) \quad \begin{aligned} f_{00}(\zeta) &= \cosh \frac{\zeta}{\lambda_0} - (\alpha_{\bar{0}} - \alpha_0) \sinh \frac{\zeta}{\lambda_0} + \sum_s \alpha_s \exp \left[\frac{\zeta}{\lambda_s} \right], \\ &= \cosh \frac{\zeta}{\lambda_0} - \frac{\lambda_0}{\delta} \sinh \frac{\zeta}{\lambda_0} + \sum_s \alpha_s \exp \left[\frac{\zeta}{\lambda_s} \right], \end{aligned}$$

with the *linear extrapolation distance* $\delta = \lambda_0/(\alpha_0 - \alpha_0)$. The *distance of the extrapolated endpoint* Δ is related to δ by the equation

$$(20) \quad \Delta = \frac{\lambda_0}{2} \log \frac{\lambda_0 + \delta}{\lambda_0 - \delta}.$$

An alternate representation of $f_{00}(\zeta)$ is therefore

$$(21) \quad f_{00}(\zeta) = \frac{\sinh (\Delta - \zeta)/\lambda_0}{\sinh \Delta/\lambda_0} + \sum_s \alpha_s \exp [\zeta/\lambda_s] \quad (\zeta \leq 0).$$

δ and Δ are dimensionless quantities which are to be multiplied by Σ^{-1} in order to obtain the extrapolation distances in centimeters. The \sum_s is the transient part, which does not contribute to the asymptotic behavior at $z = -\infty$, because the λ_s are smaller than λ_0 .

5. - A short comparison with the Gauss quadrature method ⁽⁵⁾.

The characteristic equation in the spherical harmonics method was shown to be $(1 + a\lambda_k^2)P_{L+1}(\lambda_k) - \lambda_k(c + a\lambda_k^2)W_L(\lambda_k) = 0$. $W_L(\lambda_k)$ may be represented ⁽⁶⁾ by

$$(22) \quad W_L(\lambda_k) = \frac{1}{2} \int_{-1}^{+1} \frac{P_{L+1}(\lambda_k) - P_{L+1}(u)}{\lambda_k - u} du.$$

The integrand is a polynomial of degree L in u ; hence the integral is given correctly by the Gauss quadrature method, which is here used only as a mathematical aid to give another *correct* expression for $W_L(\lambda_k)$:

$$(23) \quad W_L(\lambda_k) = \frac{1}{2} \sum_{j=\pm 1}^{\pm(L+1)/2} a_j \frac{P_{L+1}(\lambda_k) - P_{L+1}(\mu_j)}{\lambda_k - \mu_j} = \frac{1}{2} \sum_{j=\pm 1}^{\pm(L+1)/2} a_j \frac{P_{L+1}(\lambda_k)}{\lambda_k - \mu_j},$$

— because $P_{L+1}(\mu_j) = 0$ —, where

$$a_j = \frac{1}{P'_{L+1}(\mu_j)} \int_{-1}^{+1} \frac{P_{L+1}(u) du}{u - \mu_j} = -2 \frac{Q_{L+1}(\mu_j)}{P'_{L+1}(\mu_j)} = 2 \frac{W_L(\mu_j)}{P'_{L+1}(\mu_j)}.$$

Putting the last value of a_j in equation (23) one obtains the formula

$$W_L(\lambda_k) = P_{L+1}(\lambda_k) \sum_{j=\pm 1}^{\pm(L+1)/2} \frac{W_L(\mu_j)}{(\lambda_k - \mu_j)P'_{L+1}(\mu_j)},$$

⁽⁵⁾ A similar comparison has been made by J. C. MARK, *l.c.* ⁽¹⁾.

⁽⁶⁾ A. ERDÉLY *et al.*: *Higher Transcendental Functions*, Vol. I, p. 154, Eq. (31).

which could be found by an application of the Lagrange interpolation formula also. With equation (23) the characteristic equation may be written in the form

$$\left[1 + a\lambda_k^2 - \lambda_k(c + a\lambda_k^2) \frac{1}{2} \sum_{j=\pm 1}^{\pm(L+1)/2} \frac{a_j}{\lambda_k - \mu_j} \right] P_{L+1}(\lambda_k) = 0,$$

where the factor $P_{L+1}(\lambda_k)$ is different from zero as long as $c \neq 0$, $a \neq 0$. Hence, the characteristic equation of the spherical harmonics method is also

$$(24) \quad 1 + a\lambda_k^2 - \lambda_k(c + a\lambda_k^2) \frac{1}{2} \sum_{j=\pm 1}^{\pm(L+1)/2} \frac{a_j}{\lambda_k - \mu_j} = 0,$$

or, summing over positive j 's only and using $a_{-j} = a_j$, $\mu_{-j} = -\mu_j$

$$\sum_{j=1}^{(L+1)/2} \frac{a_j}{\lambda_k^2 - \mu_j^2} = \frac{1 + a\lambda_k^2}{\lambda_k^2(c + a\lambda_k^2)}.$$

The same equation will now be derived with the C-W-method. The original Boltzmann equation was

$$\mu \frac{\partial f(\zeta, \mu)}{\partial \zeta} + f(\zeta, \mu) = \frac{1}{2} (1 - \gamma_a) \int_{-1}^{+1} f(\zeta, \mu') d\mu' + \frac{3}{2} (1 - \gamma_t) \mu \int_{-1}^{+1} \mu' f(\zeta, \mu') d\mu'.$$

Integrating this equation over μ from -1 to $+1$ one gets

$$\frac{\partial}{\partial \zeta} \int_{-1}^{+1} \mu f(\zeta, \mu) d\mu + \int_{-1}^{+1} f(\zeta, \mu) d\mu = \frac{1}{2} (1 - \gamma_a) \int_{-1}^{+1} f(\zeta, \mu') d\mu' \int_{-1}^{+1} d\mu + \frac{3}{2} (1 - \gamma_t) \int_{-1}^{+1} \mu' f(\zeta, \mu') d\mu' \int_{-1}^{+1} \mu d\mu.$$

The last integral over μ vanishes and one finds the relation

$$(25) \quad \frac{\partial}{\partial \zeta} \int_{-1}^{+1} \mu f(\zeta, \mu) d\mu = -\gamma_a \int_{-1}^{+1} f(\zeta, \mu) d\mu,$$

between the two integrals on the right side of the Boltzmann equation. This is an essential equation for solving the anisotropic case with the C-W-method. If one assumes now a partial solution with an arbitrary constant λ_r

$$f(\zeta, \mu) \sim g(\mu) \exp[-\zeta/\lambda_r]$$

one has

$$\frac{\partial f}{\partial \zeta} \sim -\frac{1}{\lambda_r} g(\mu) \exp[-\zeta/\lambda_r]$$

and from equation (25)

$$\int_{-1}^{+1} \mu f(\zeta, \mu) d\mu = \gamma_a \lambda_r \int_{-1}^{+1} f(\zeta, \mu) d\mu.$$

Hence, the Boltzmann equation yields

$$(26) \quad \left(-\frac{\mu}{\lambda_r} + 1\right) g(\mu) = \left[\frac{1}{2}(1 - \gamma_a) + \frac{3}{2}\gamma_a(1 - \gamma_t)\lambda_r\mu\right] \int_{-1}^{+1} g(\mu) d\mu = \frac{1}{2}(c + a\lambda_r\mu)C,$$

with

$$C = \int_{-1}^{+1} g(\mu) d\mu = \text{const}$$

and one obtains the solution

$$(27) \quad g(\mu) = \frac{C}{2} \frac{\lambda_r}{\lambda_r - \mu} (c + a\lambda_r\mu)$$

as a μ -dependent amplitude of the partial solution $f(\zeta, \mu)$ above. The expression (27) for $g(\mu)$ shows a pole at $\mu = \lambda_r$. In conformity with the physical conditions of the Milne problem one supposes therefore $g(\mu) = 0$ for positive μ and equal to (27) for negative μ at $\zeta = 0$. Equations (26) and (27) are consistent only for certain λ_r 's, which one gets by carrying out the integral

$$C = \int_{-1}^{+1} g(\mu) d\mu = \frac{C}{2} \int_{-1}^0 \frac{\lambda_r(c + a\lambda_r\mu)}{\lambda_r - \mu} d\mu.$$

Hence, it follows a transcendental characteristic equation for the λ_r 's:

$$(27a) \quad 1 + a\lambda_r^2 - \lambda_r(c + a\lambda_r^2) \log \frac{1 + \lambda_r}{\lambda_r} = -1.$$

Selecting the λ_r 's as roots of this equation, C in (27) may be considered as an arbitrary constant. For $\zeta \neq 0$, however, the backward scattering has to be considered and the divergence of the integral would become manifest.

Therefore, one replaces the integral equation (26) by a system of linear equations, evaluating the integral in *L*-approximation by Gauss quadrature method and using instead of the λ_r of equation (27a) another λ_k

$$(28) \quad \left(-\frac{\mu_i}{\lambda_k} + 1\right) g(\mu_i) = \frac{1}{2}(c + a\lambda_k\mu_i) \sum_{j=\pm 1}^{\pm(L+1)/2} a_j g(\mu_j) \quad (i = \pm 1, \pm 2, \dots, \pm(L+1)/2)$$

The μ_j 's are also in the Gauss quadrature method the roots of $P_{L+1}(\mu_j) = 0$ and we have here $L+1$ linear equations for the determination of the $g(\mu_j)$ which depend on λ_k by equation (27). Therefore, one obtains by introduction of $g(\mu)$ from equation (27) in the equation (28) the characteristic equation of the C-W-method for those possible λ_k , for which the system of linear equations (28) has a solution. This gives

$$\begin{aligned} 1 &= \frac{1}{2} \sum_{i=\pm 1}^{\pm(L+1)/2} a_j \frac{\lambda_k}{\lambda_k - \mu_j} (c + a\lambda_k\mu_j) = \frac{1}{2} \sum_{j=\pm 1}^{\pm(L+1)/2} a_j \lambda_k \left[-a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k - \mu_j} \right] = \\ &= -\frac{a}{2} \lambda_k^2 \sum_{j=\pm 1}^{\pm(L+1)/2} a_j + \frac{1}{2} \lambda_k (c + a\lambda_k^2) \sum_{j=\pm 1}^{\pm(L+1)/2} \frac{a_j}{\lambda_k - \mu_j}; \end{aligned}$$

because $\sum_{j=\pm 1}^{\pm(L+1)/2} a_j = 2$, we get finally the characteristic equation in the C-W method

$$(29) \quad 1 + a\lambda_k^2 - \lambda_k(c + a\lambda_k^2) \frac{1}{2} \sum_{j=\pm 1}^{\pm(L+1)/2} \frac{a_j}{\lambda_k - \mu_j} = 0$$

which is identical with equation (24). Hence their roots λ_k are identical also.

In the C-W method (7) one has now to assume a superposition of partial solutions—for the negative z of our medium with the opposite sign of the λ_k 's—

$$(30) \quad f_{\text{C-W}}(z, \mu) = \sum_k g_{-k}(\mu) \exp[\zeta/\lambda_k] = \sum_k \alpha_k \lambda_k \frac{c - a\lambda_k \mu}{\lambda_k + \mu} \exp[\zeta/\lambda_k] \equiv \\ \equiv \sum_k \alpha_k \lambda_k \exp[\zeta/\lambda_k] \left[-a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k + \mu} \right],$$

where the α_k 's replace the constants $\frac{1}{2}C_k$'s of equation (27) for the set of $-\lambda_k$'s; as before they must be determined from the boundary condition at $z=0$. One recognizes the identity of the C-W expression, (30) and the first part $f^1(z, \mu)$ of the spherical harmonics method equation (17).

6. - Completion of the solution.

For the fluxes at the boundary $z=0$ we have two expressions: equation (16) on the medium side and equation (13) on the vacuum side

$$(31) \quad f(0, \mu) = \sum_k \alpha_k \lambda_k \left\{ -a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k + \mu} \left[1 - \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \right] \right\} \\ = (L+1)P_{L+1}(\mu) \sum_{j=1}^{(L+1)/2} \beta_j \frac{P_L(\mu_j)}{\mu - \mu_j}.$$

They must be equal for every μ between $-1 \leq \mu \leq 1$; it can be shown by a suitable application of the Lagrange interpolation formula to the medium side of this equation, that the β_j 's can always be chosen in such a way to fit this condition. To determine the remaining $(L+1)/2$ α_k 's (this means for instance $\alpha_0 - \alpha_0 = \lambda_0/\delta$, α_1 , α_2 , ..., $\alpha_{(L-1)/2}$) it is sufficient to remark that the vacuum side expression vanishes at $\mu = -\mu_j$. Hence the medium side expression has to vanish at the same points; this yields $(L+1)/2$ inhomogeneous linear equations for the remaining $(L+1)/2$ α_k 's:

$$(32) \quad \sum_k \alpha_k \lambda_k \left\{ -a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k - \mu_j} \right\} = 0 \quad \left(j = 1, 2, \dots, \frac{L+1}{2} \right).$$

(7) S. CHANDRASEKHAR: *Radiative Transfer* (Oxford, 1950).

Because $P_{L+1}(-\mu_j) = 0$ in equation (31) the C-W method and spherical harmonics method lead to the same equations (32) for the coefficients α_k . The appearance of the μ_j 's in both methods is a special feature of the Milne problem and other problems with a vacuum on one side. In these cases the replacement of the vacuum by a perfect absorber leads in P_L approximation to the use of the pure Legendre polynomials of the 1st kind on the vacuum side, whereas they are introduced as a mathematical substitute in the C-W method on the medium side.

The calculation of the coefficients α_k is carried out in the Appendix II. In solving the system (32) of linear equations one is led to the introduction of « half Legendre polynomials » which have only the positive roots μ_j of (14) and λ_s ($s = 1, 2, \dots, (L-1)/2$) of (12), namely

$$(33) \quad p(\lambda) = \prod_{j=1}^{(L+1)/2} (\lambda - \mu_j) \quad \text{and} \quad r(\lambda) = \prod_{s=1}^{(L-1)/2} (\lambda - \lambda_s)$$

excluding the largest root λ_0 of (12) from $r(\lambda)$. Intimately connected with the extrapolation distances δ and Δ is the following function of λ

$$(34) \quad \delta(\lambda) = \lambda \frac{r(\lambda)p(-\lambda) + r(-\lambda)p(\lambda)}{r(\lambda)p(-\lambda) - r(-\lambda)p(\lambda)},$$

$\delta(\lambda)$ is a function of λ^2 only. One may use this function for a shorter representation of the coefficients α_k .

After introducing two further abbreviations

$$(35) \quad \lambda_a = i \sqrt{\frac{c}{a}},$$

and for a common factor of the α_k 's independent of k

$$(36) \quad N = \frac{r(\lambda_0)r(-\lambda_0)}{r(\lambda_0)p(-\lambda_0) + r(-\lambda_0)p(\lambda_0)} \frac{\delta(\lambda_0)}{\delta(\lambda_0) + (a/c)\lambda_0^2\delta(\lambda_a)},$$

the solution α_k of equation (32) may be written

$$(37) \quad \alpha_0 = N \left[1 + \frac{a}{c} \lambda_0 \delta(\lambda_a) \right] \frac{p(-\lambda_0)}{r(-\lambda_0)},$$

$$(38) \quad \alpha_0 = N \left[1 - \frac{a}{c} \lambda_0 \delta(\lambda_a) \right] \frac{p(\lambda_0)}{r(\lambda_0)},$$

$$(39) \quad \alpha_s = N \left[1 - \frac{a}{c} \lambda_s \delta(\lambda_a) \right] \frac{c + a\lambda_s^2}{c + a\lambda_s^2} \frac{2\lambda_0^2 p(\lambda_s)}{\lambda_s(\lambda_s^2 - \lambda_0^2)r'(\lambda_s)} \quad \left(s = 1, 2, \dots, \frac{L-1}{2} \right).$$

The extrapolation distances are—see equations (19) and (20)—

$$(40) \quad \delta = \frac{\delta(\lambda_0) + (a/c)\lambda_0^2 \delta(\lambda_a)}{1 + (a/c)\delta(\lambda_a)\delta(\lambda_0)} \quad \text{with} \quad \begin{cases} c = 1 - \gamma_a, \\ a = 3\gamma_a(1 - \gamma_t), \end{cases}$$

and $[\text{tgh}^{-1}$ stands for the *inverse function* of the hyperbolic tangent]

$$\Delta = \lambda_0 \left[\text{tgh}^{-1} \frac{\delta(\lambda_0)}{\lambda_0} + \text{tgh}^{-1} \frac{a}{c} \lambda_0 \delta(\lambda_a) \right].$$

Introducing the coefficients α_k in equation (16) one obtains the solution for the directed flux $f(z, \mu)$ in every direction $\mu = \cos \vartheta$ and on every plane $z = \zeta/\Sigma$ of the medium in P_L -approximation for the Milne problem with anisotropic scattering and absorption. The case of fission, with production of neutrons having the same energy as the neutrons which induced the fission, is covered by the formalism by putting $\gamma_a < 0$.

At the boundary $z = 0$ the 1st part of $f(0, \mu)$ namely $f^I(0, \mu)$, *i.e.* the C-W part for positive μ —equation (17) with the coefficients (37)–(39)—can be summed up to a closed expression by a calculation carried out in the Appendix III.

$$(41) \quad f^I(0, \mu) = 2N(c + a\lambda_0^2) \left[1 + \frac{a}{c} \delta(\lambda_a)\mu \right] \frac{\lambda_0^2}{\lambda_0^2 - \mu^2} \frac{p(-\mu)}{r(-\mu)}.$$

This expression has poles at $\mu = -\lambda_0$. Remembering that $f(0, \mu)$ is a polynomial of degree L in μ coinciding with $f^I(0, \mu)$ at the points $\mu = \mu_j$ and having zeros at the points $\mu = -\mu_j$ one is able to apply the Lagrange interpolation formula to construct this polynomial, which represents the directed flux at the boundary according to the *spherical harmonics method* and which is regular for all μ ($-1 \leq \mu \leq 1$):

$$(42) \quad f(0, \mu) = \sum_{j=1}^{(L+1)/2} \frac{f^I(0, \mu_j) P_{L+1}(\mu)}{(\mu - \mu_j) P'_{L+1}(\mu_j)} = \\ = 2N\lambda_0^2(c + a\lambda_0^2)p(-\mu)p(\mu) \sum_{j=1}^{(L+1)/2} \frac{1 + (a/c)\delta(\lambda_a)\mu_j}{(\mu - \mu_j)(\lambda_0^2 - \mu_j^2)r(-\mu_j)p'(\mu_j)}.$$

By comparison of equation (42) with the second expression of equation (13) for $\zeta = 0$ finally the coefficients β_j of the vacuum side representation are obtained

$$(43) \quad \beta_j = \frac{f^I(0, \mu_j)}{(L+1)P_L(\mu_j)P'_{L+1}(\mu)} = \frac{2N\lambda_0^2(c + a\lambda_0^2)[1 + (a/c)\delta(\lambda_a)\mu_j]p(-\mu_j)}{(L+1)P_L(\mu_j)P'_{L+1}(\mu_j)(\lambda_0^2 - \mu_j^2)r(-\mu_j)}.$$

S. CHANDRASEKHAR introduced in his book *Radiative Transfer*, p. 127, a function $H(\mu)$, to which the functions $p(\mu)$ and $r(\mu)$ here are related by the

equation

$$H(\mu) = \frac{\lambda_0 \lambda_1 \dots \lambda_{(L-1)/2}}{\mu_1 \mu_2 \dots \mu_{(L+1)/2}} \frac{-p(-\mu)}{(\lambda_0 + \mu)r(-\mu)} = \frac{-1}{\sqrt{1-c-(a/3)}} \frac{p(-\mu)}{(\lambda_0 + \mu)r(-\mu)} =$$

$$= \frac{\lambda_0}{\lambda_0 + \mu} \frac{r(0)}{p(0)} \frac{p(-\mu)}{r(-\mu)}. \quad \left(\text{Remark: } 1-c-\frac{a}{3} = \gamma_a \gamma_t \right)$$

7. - Investigation of $f^{II}(0, \mu)$.

The representation of $f(0, \mu)$ given in equation (42) has the form of equation (13) for $z=0$, or of the second on the right side of equation (31), both of which were derived from a consideration of the vacuum side. Therefore it was possible also to deduce from formula (42) the coefficients β_j (equation (43)) of the vacuum side representation of $f(0, \mu)$. We obtain the medium side representation if we put the α_k 's of equations (37)–(39) in the first term on the right side of equation (31) or in equation (16) for $z=0$. Then we get the medium side representation, which is rather lengthy.

The connection between medium side and vacuum side representation is quickly obtained by an application of the Lagrange interpolation formula to the medium side representation of $f^{II}(0, \mu)$. To this end we construct the function $f(\lambda) = 1 - \beta\lambda$ with the use of its values at the $L+2$ points $\lambda_j = (-\lambda_0, \lambda_0, \lambda_1, \dots, \lambda_{(L-1)/2}, -\mu_1 - \mu_2, \dots, -\mu_{(L+1)/2})$ at which the polynomial $R(\lambda) = (\lambda^2 - \lambda_0^2)r(\lambda)p(-\lambda)$ has its zeros. Then the Lagrange interpolation formula

$$(44) \quad \frac{f(\lambda)}{R(\lambda)} = \sum_{\text{all } \lambda_j} \frac{f(\lambda_j)}{(\lambda - \lambda_j) R'(\lambda_j)},$$

gives for $\lambda = -\mu$

$$(45) \quad \frac{1 + \beta\mu}{(\mu^2 - \lambda_0^2)r(-\mu)p(\mu)} = - \left\{ \frac{1 + \beta\lambda_0}{(\lambda_0 - \mu)2\lambda_0 r(-\lambda_0)p(\lambda_0)} + \frac{1 - \beta\lambda_0}{(\lambda_0 + \mu)2\lambda_0 r(\lambda_0)p(-\lambda_0)} + \right.$$

$$\left. + \sum_{s=1}^{(L-1)/2} \frac{1 - \beta\lambda_s}{(\lambda_s + \mu)(\lambda_s^2 - \lambda_0^2)r'(\lambda_s)p(-\lambda_s)} + \sum_{j=1}^{(L+1)/2} \frac{1 + \beta\mu_j}{(\mu - \mu_j)(\lambda_0^2 - \mu_j^2)r(-\mu_j)p'(\mu_j)} \right\}.$$

Now $f^{II}(0, \mu)$ in medium side representation (see equation (18))

$$f^{II}(0, \mu) = - \sum_k \alpha_k \lambda_k \frac{c + a\lambda_k^2}{\lambda_k + \mu} \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} = - \sum_k \alpha_k \lambda_k \frac{c + a\lambda_k^2}{\lambda_k + \mu} \frac{p(\mu)p(-\mu)}{p(\lambda_k)p(-\lambda_k)},$$

with the coefficients α_k from equations (37)–(39)

$$(46) \quad = -N\lambda_0(c + a\lambda_0^2)p(\mu)p(-\mu) \left\{ \frac{1 + (a/c)\delta(\lambda_a)\lambda_0}{(\lambda_0 - \mu)r(-\lambda_0)p(\lambda_0)} + \frac{1 - (a/c)\delta(\lambda_a)\lambda_0}{(\lambda_0 + \mu)r(\lambda_0)p(-\lambda_0)} + \right.$$

$$\left. + 2\lambda_0 \sum_{s=1}^{(L-1)/2} \frac{1 - (a/c)\delta(\lambda_a)\lambda_s}{(\lambda_s + \mu)(\lambda_s^2 - \lambda_0^2)r'(\lambda_s)p(-\lambda_s)} \right\}$$

is simply proportional to the first 3 terms on the right side of equation (45) with the constant $\beta = (a/c)\delta(\lambda_a)$. Hence by comparison with equation (45) we may write $f^{II}(0, \mu)$

of equation (46) also in the following from

$$\begin{aligned}
 (47) \quad f^{II}(0, \mu) &= N \cdot 2\lambda_0^2(c + a\lambda_0^2) p(\mu) p(-\mu) \cdot \\
 &\cdot \left\{ \frac{1 + (a/c) \delta(\lambda_a) \mu}{(\mu^2 - \lambda_0^2) r(-\mu) p(\mu)} + \sum_{j=1}^{(L+1)/2} \frac{1 + (a/c) \delta(\lambda_a) \mu_j}{(\mu - \mu_j)(\lambda_0^2 - \mu_j^2) r(-\mu_j) p'(\mu_j)} \right\} = \\
 &= -2N(c + a\lambda_0^2) \left[1 + \frac{a}{c} \delta(\lambda_a) \mu \right] \frac{\lambda_0^2}{\lambda_0^2 - \mu^2} \frac{p(-\mu)}{r(-\mu)} + \\
 &+ 2N\lambda_0^2(c + a\lambda_0^2) p(\mu) p(-\mu) \sum_{j=1}^{(L+1)/2} \frac{1 + (a/c) \delta(\lambda_a) \mu_j}{(\mu - \mu_j)(\lambda_0^2 - \mu_j^2) r(-\mu_j) p'(\mu_j)}.
 \end{aligned}$$

The first term on the right side of the last equation is $-f^I(0, \mu)$ in the form of equation (41) again. Hence we have extracted by the use of the Lagrange formula the « pole-part » $-f^I(0, \mu)$ of $f^{II}(0, \mu)$. The second term is the result of the spherical harmonics method $f(0, \mu)$ in the vacuum side representation of equation (42) again. Keeping equation (46) and (47) together we have the relation between the two representations of $f(0, \mu)$ on the medium side (M.S.) and on the vacuum side (V.S.)

$$\begin{aligned}
 (48) \quad f(0, \mu) &= N \cdot 2\lambda_0^2(c + a\lambda_0^2) \left\{ \frac{1 + (a/c) \delta(\lambda_a) \mu}{\lambda_0^2 - \mu^2} \frac{p(-\mu)}{r(-\mu)} + \right. \\
 &+ p(\mu) p(-\mu) \left[\frac{1 + (a/c) \delta(\lambda_a) \lambda_0}{2(\lambda_0 - \mu) r(-\lambda_0) p(\lambda_0)} + \frac{1 - (a/c) \delta(\lambda_a) \lambda_0}{2(\lambda_0 + \mu) r(\lambda_0) p(-\lambda_0)} + \right. \\
 &\left. \left. + \sum_{s=1}^{(L-1)/2} \frac{1 - (a/c) \delta(\lambda_a) \lambda_s}{(\lambda_s + \mu)(\lambda_s^2 - \lambda_0^2) r'(\lambda_s) p(-\lambda_s)} \right] \right\} \quad (\text{M.S.})
 \end{aligned}$$

$$(49) \quad = N \cdot 2\lambda_0^2(c + a\lambda_0^2) p(\mu) p(-\mu) \sum_{j=1}^{(L+1)/2} \frac{1 + (a/c) \delta(\lambda_a) \mu_j}{(\mu - \mu_j)(\lambda_0^2 - \mu_j^2) r(-\mu_j) p'(\mu_j)} \quad (\text{V.S.}).$$

8. - Density and current.

In our monoenergetic treatment of the problems we may assume that our velocity is the unit-velocity. Then we do not discriminate formulas for flux and density etc.

We may replace $\lambda_k(c + a\lambda_k^2)$ in equation (16) with the help of characteristic equation (12)

$$\lambda_k(c + a\lambda_k^2) = \frac{(1 + a\lambda_k^2) P_{L+1}(\lambda_k)}{W_L(\lambda_k)}$$

and obtain another sometimes convenient formula for $f(z, \mu)$, $z \leq 0$, L odd:

$$(50) \quad f(z, \mu) = \sum_k \alpha_k \exp[\xi/\lambda_k] \left\{ -a\lambda_k^2 + \frac{1 + a\lambda_k^2}{W_L(\lambda_k)} \frac{P_{L+1}(\lambda_k) - P_{L+1}(-\mu)}{\lambda_k + \mu} \right\}.$$

Because

$$W_L(\lambda_k) = \frac{1}{2} \int_{-1}^{+1} \frac{P_{L+1}(\lambda_k) - P_{L+1}(-\mu)}{\lambda_k + \mu} d\mu,$$

we obtain quickly from (50) the *density* ($z \leq 0$)

$$\begin{aligned}
 (51) \quad \varrho(z) &= 2\pi \int_{-1}^{+1} f(\zeta, \mu) d\mu = 4\pi \sum_k \alpha_k \exp[\zeta/\lambda_k] \cdot \\
 &\quad \cdot \left\{ -a\lambda_k^2 \frac{1}{2} \int_{-1}^{+1} d\mu + \frac{1+a\lambda_k^2}{W_L(\lambda_k)} \frac{1}{2} \int_{-1}^{+1} \frac{P_{L+1}(\lambda_k) - P_{L+1}(-\mu)}{\lambda_k + \mu} d\mu \right\} = \\
 &= 4\pi \sum_k \alpha_k \exp[\zeta/\lambda_k] = 4\pi f_{00}(\zeta) = \frac{4\pi}{c} f^i(z, 0).
 \end{aligned}$$

At the boundary $z=0$ the density $\varrho(0) = 4\pi \sum_k \alpha_k$ is related to the 1st part of directed flux $f^i(0, 0)$ in the direction $\mu=0$ which is according to equation (17) and equation (41)

$$(52) \quad f^i(0, 0) = c \sum_k \alpha_k = 2N(c + a\lambda_0^2) \frac{p(0)}{r(0)}.$$

Hence the density at the boundary in the spherical harmonics method is

$$(53) \quad \varrho(0) = 4\pi \sum_k \alpha_k = 8\pi N \left(1 + \frac{a}{c} \lambda_0^2 \right) \frac{p(0)}{r(0)}.$$

This equation relates also the density at $z=0$ with the normalization factor N and could be used for a change in normalization from N to unit density. The normalization with N results from the choice in equation (19).

Also the current $j(z)$ for $z \leq 0$ is quickly obtained with the help of equation (50):

$$\begin{aligned}
 (54) \quad j(z) &= 2\pi \int_{-1}^{+1} \mu f(z, \mu) d\mu, \\
 &= 4\pi \sum_k \alpha_k \exp[\zeta/\lambda_k] \left\{ -a\lambda_k^2 \cdot \frac{1}{2} \int_{-1}^{+1} \mu d\mu + \frac{1+a\lambda_k^2}{W_L(\lambda_k)} \frac{1}{2} \int_{-1}^{+1} \mu \frac{P_{L+1}(\lambda_k) - P_{L+1}(-\mu)}{\lambda_k + \mu} d\mu \right\} = \\
 &= 4\pi \sum_k \alpha_k \exp[\zeta/\lambda_k] \frac{1+a\lambda_k^2}{W_L(\lambda_k)} \cdot \frac{1}{2} \int_{-1}^{+1} d\mu \left[P_{L+1}(\lambda_k) - P_{L+1}(-\mu) - \lambda_k \frac{P_{L+1}(\lambda_k) - P_{L+1}(-\mu)}{\lambda_k + \mu} \right] = \\
 &= 4\pi \sum_k \alpha_k \exp[\zeta/\lambda_k] \frac{1+a\lambda_k^2}{W_L(\lambda_k)} [P_{L+1}(\lambda_k) - \lambda_k W_L(\lambda_k)] = \\
 &= 4\pi \sum_k \alpha_k \exp[\zeta/\lambda_k] \frac{1+a\lambda_k^2}{W_L(\lambda_k)} \left[\frac{c+a\lambda_k^2}{1+a\lambda_k^2} - 1 \right] \lambda_k W(\lambda_k), \quad \text{with (12),} \\
 &= 4\pi(c-1) \sum_k \alpha_k \lambda_k \exp[\zeta/\lambda_k] = -4\pi\gamma_a \sum_k \alpha_k \lambda_k \exp[\zeta/\lambda_k] = \frac{4\pi}{3} f_{10}(\zeta).
 \end{aligned}$$

For $z=0$ we have the current at the boundary

$$(55) \quad j(0) = 2\pi \int_{-1}^{+1} \mu f(0, \mu) d\mu = \frac{4\pi}{3} f_{10}(0) = -4\pi\gamma_a \sum_k \alpha_k \lambda_k = \\ = -\frac{16\pi N \gamma_a \lambda_0^2 p(\lambda_a) p(-\lambda_a)}{\lambda_a [r(\lambda_a) p(-\lambda_a) - r(-\lambda_a) p(\lambda_a)]}.$$

The last result is obtained by a further application of the Lagrange interpolation (see Appendix IV). Equation (55) might also be used to change the normalization to unit current in z -direction.

9. - Special cases.

a) *No absorption*: $\gamma_a = 0$.

For no absorption the quantities $c = 1 - \gamma_a \rightarrow 1$ and $a = 3\gamma_a(1 - \gamma_t) = 0$ whatever finite value γ_t may have. The first two equations of the system (3) show that in the limit $\gamma_a = 0$

$$(56) \quad f_{00}(\zeta) = 1 - \frac{\zeta}{\delta} + \sum_{s=1}^{(L-1)/2} \alpha_s \exp[\zeta/\lambda_s],$$

$$(57) \quad f_{10}(\zeta) = \frac{1}{\delta \cdot \gamma_t}.$$

This means that the asymptotic part of $f_{00}(\zeta)$ degenerates into a straight line and that $f_{10}(\zeta)$ is a constant and does not contain the transients s . The $f_{l0}(\zeta)$ for $l \geq 2$ contain instead only transients and no asymptotic part. The square of the largest root of the characteristic equation (12) may be estimated to

$$(58) \quad \lambda_0^2 \approx \frac{1 + (4/5)\gamma_a + (108/175)\gamma_a^2\gamma_t}{3\gamma_a\gamma_t},$$

in the general case for small γ_a and $1 - \gamma_t$. In the limit $\gamma_a = 0$ and arbitrary γ_t this expression goes to infinity, or precisely

$$(59) \quad 3\gamma_a\gamma_t\lambda_0^2 \rightarrow 1.$$

Also $\lambda_a = i\sqrt{c/a}$ goes to infinity $\lambda_a \rightarrow i\infty$:

$$\frac{a}{c} \lambda_0^2 = \frac{3\gamma_a(1 - \gamma_t)}{1 - \gamma_a} \lambda_0^2 \rightarrow \frac{1 - \gamma_t}{\gamma_t}; \quad 1 + \frac{a}{c} \lambda_0^2 \rightarrow \frac{1}{\gamma_t}.$$

A development of $\delta(\lambda)$ in a power series of $1/\lambda$ shows

$$(60) \quad \delta(\lambda) = M_1 - A_1 + \frac{1}{\lambda^2} [M_3 - M_2 M_1 + M_1 A_1 (M_1 - A_1) + A_1 A_2 - A_3] + \\ + \frac{1}{\lambda^4} (\dots) + \dots,$$

where

$$(61) \quad \left\{ \begin{array}{lll} M_1 = \sum_{j=1}^{(L+1)/2} \mu_j, & M_2 = \sum_{\substack{j,k \\ j < k}}^{(L+1)/2} \mu_j \mu_k, & M_3 = \sum_{\substack{j,k,l \\ j < k < l}}^{(L+1)/2} \mu_j \mu_k \mu_l, \dots \\ A_1 = \sum_{s=1}^{(L-1)/2} \lambda_s, & A_2 = \sum_{\substack{s,t \\ s < t}}^{(L-1)/2} \lambda_s \lambda_t, & A_3 = \sum_{\substack{s,t,u \\ s < t < u}}^{(L-1)/2} \lambda_s \lambda_t \lambda_u, \dots \end{array} \right.$$

are the primitive symmetrical functions of the roots μ_j and λ_s . For $|\lambda| = \infty$ only the 1st term of the series (60) remains; hence we have in the special case $\gamma_a = 0$, $\lambda_0 \rightarrow \infty$, $\lambda_a \rightarrow i\infty$

$$(62) \quad \delta(\lambda_0) = \delta(\lambda_a) \rightarrow M_1 - A_1,$$

$$(63) \quad \frac{r(\lambda_0)r(-\lambda_0)}{r(\lambda_0)p(-\lambda_0) + r(-\lambda_0)p(\lambda_0)} \rightarrow \frac{-1}{2(M_1 - A_1)}, \quad N \rightarrow -\frac{\gamma_t}{2(M_1 - A_1)}.$$

The two extrapolated distances become equal: $\delta = \Delta \rightarrow (M_1 - A_1)/\gamma_t$; the coefficients $\alpha_{\bar{0}}$ and α_0 diverge, but in the development of the first term of $f_{00}(\zeta)$

$$\alpha_{\bar{0}} \exp[-\zeta/\lambda_0] + \alpha_0 \exp[\zeta/\lambda_0] = \\ = \alpha_{\bar{0}} + \alpha_0 - \frac{\alpha_{\bar{0}} - \alpha_0}{\lambda_0} \zeta + \frac{1}{2} \frac{1}{\lambda_0^2} (\alpha_{\bar{0}} + \alpha_0) \zeta^2 + \dots \rightarrow 1 - \frac{\alpha_{\bar{0}} - \alpha_0}{\lambda_0} \zeta,$$

only the sums $\alpha_{\bar{0}} + \alpha_0 = 1$ and $(\alpha_{\bar{0}} - \alpha_0)/\lambda_0 \rightarrow \gamma_t/(M_1 - A_1) = 1/\delta$ play a role. In this way one gets systematically equation (56) again

$$f_{00}(\zeta) \rightarrow 1 - \frac{\zeta}{\delta} + \sum_s \alpha_s \exp[\zeta/\lambda_s] = 1 - \frac{\gamma_t \zeta}{M_1 - A_1} + \sum_s \alpha_s \exp[\zeta/\lambda_s].$$

For

$$f_{10}(\zeta) = 3\gamma_a \{ \alpha_{\bar{0}} \lambda_0 \exp[-\zeta/\lambda_0] - \alpha_0 \lambda_0 \exp[\zeta/\lambda_0] \} - 3\gamma_a \sum_s \alpha_s \lambda_s \exp[\zeta/\lambda_s] = \\ = \left\{ 3\gamma_a \lambda_0 (\alpha_{\bar{0}} - \alpha_0) - 3\gamma_a \zeta + \frac{3\gamma_a}{2\lambda_0} (\alpha_{\bar{0}} - \alpha_0) \zeta^2 + \dots \right\} - 3\gamma_a \sum_s \alpha_s \lambda_s \exp[\zeta/\lambda_s]$$

only the first term remains after the limiting process

$$f_{10}(\xi) \rightarrow 3\gamma_a \lambda_0^2 \frac{\alpha_0 - \alpha_0}{\lambda_0} \rightarrow \frac{1}{\delta \cdot \gamma_t} = \frac{1}{M_1 - A_1},$$

which is equation (57) again. The asymptotic part plays no role in the higher f_{i0} 's, but the coefficients α_s of the transients behave regularly since

$$(64) \quad \alpha_s \rightarrow \frac{p(\lambda_s)}{(M_1 - A_1) \lambda_s r'(\lambda_s)}.$$

The characteristic equation (12) is now only of the degree $L-1$, because 2 roots $\lambda_0, -\lambda_0$ are infinite, and (12) becomes

$$(65) \quad P_{L+1}(\lambda_s) - \lambda_s W_L(\lambda_s) = 0 \quad (\pm \lambda_1, \pm \lambda_2, \dots, \pm \lambda_{(L-1)/2}).$$

(As in (12), the largest coefficient in this equation vanishes.) One recognizes in (65) the independence of the λ_s of γ_t , and (64) shows that the α_s are also independent of γ_t . Only $\delta = A$ depends on γ_t . The same is the case with directed fluxes through the boundary. The Gauss quadrature method gives

$$(66) \quad f(0, \mu) = \frac{-p(-\mu)}{(M_1 - A_1)r(-\mu)}, \quad (\mu \geq 0),$$

(the $-$ sign compensates one $-$ sign in $p(-\mu)$, which contains one (-1) factor more than $r(-\mu)$), and the spherical harmonics method

$$(67) \quad f(0, \mu) = -\frac{p(-\mu)p(\mu)}{M_1 - A_1} \sum_{j=1}^{(L+1)/2} \frac{1}{(\mu - \mu_j)r(-\mu_j)p'(\mu_j)}.$$

b) Isotropic scattering, absorption: $\gamma_a \neq 0$, $\gamma_t = 1$. (L finite).

In this case $c = 1 - \gamma_a$ is different from 1, but $a = 3\gamma_a(1 - \gamma_t) = 0$. The root λ_0 of (12) will remain finite $\lambda_0^2 \approx (1 + \frac{4}{3}\gamma_a)/3\gamma_a$ (see equation (58)). $\lambda_a = i\sqrt{c/a}$ however, is still infinite as $\lambda_a \rightarrow i\infty$. Hence one obtains by equation (62) $\delta(\lambda_a) = M_1 - A_1$. The other quantities become

$$(68) \quad \left\{ \begin{array}{l} \frac{a}{c} \lambda_0^2 \rightarrow 0, \quad N \rightarrow N_1 = \frac{r(\lambda_0)r(-\lambda_0)}{r(\lambda_0)p(-\lambda_0) + r(-\lambda_0)p(\lambda_0)}, \\ \alpha_0 = N_1 \frac{p(-\lambda_0)}{r(-\lambda_0)}, \quad \alpha_0 = N_1 \frac{p(\lambda_0)}{r(\lambda_0)}, \quad \alpha_s = N_1 \frac{2\lambda_0^2 p(\lambda_s)}{\lambda_s(\lambda_s^2 - \lambda_0^2)r'(\lambda_s)}, \\ \delta = \delta(\lambda_0), \quad A = \lambda_0 \operatorname{tgh}^{-1} \frac{\delta(\lambda_0)}{\lambda_0}, \quad \left(s = 1, 2, \dots, \frac{L-1}{2} \right). \end{array} \right.$$

The directed flux at the boundary according to the Gauss quadrature method is

$$(69) \quad f(0, \mu) = 2N_1 e \frac{\lambda_0^2}{\lambda_0^2 - \mu^2} \frac{p(-\mu)}{r(-\mu)}, \quad (\mu \geq 0),$$

and according to the spherical harmonics method

$$(70) \quad f(0, \mu) = 2N_1 e \lambda_0^2 p(-\mu) p(\mu) \sum_{j=1}^{(L+1)/2} \frac{1}{(\mu - \mu_j)(\lambda_0^2 - \mu_j^2) r(-\mu_j) p'(\mu_j)}.$$

APPENDIX I

Calculation of the sum over l in equation (15).

With $W_{l-1} = P_l Q_0 - Q_l$ we have (4)

$$\begin{aligned} & \sum_{l=2}^L (2l+1) [(1 + a\lambda_k^2) P_l(-\lambda_k) + \lambda_k(c + a\lambda_k^2) W_{l-1}(-\lambda_k)] P_l(\mu) = \\ &= \sum_{l=2}^L (2l+1) \{ [1 + a\lambda_k^2 + \lambda_k(c + a\lambda_k^2) Q_0(-\lambda_k)] P_l(-\lambda_k) P_l(\mu) - \\ & \quad - \lambda_k(c + a\lambda_k^2) Q_l(-\lambda_k) P_l(\mu) \} = \\ &= \sum_{l=0}^L (2l+1) \{ \dots \} - \\ & - [(1 + a\lambda_k^2 + \lambda_k(c + a\lambda_k^2) Q_0(-\lambda_k))(1 - 3\lambda_k \mu) + \lambda_k(c + a\lambda_k^2)(Q_0(-\lambda_k) + 3\mu Q_1(-\lambda_k))] = \\ &= \left(\sum_{l=0}^L (2l+1) \{ \dots \} \right) - 1 + 3\gamma_a \lambda_k \mu - a\lambda_k^2 \quad \text{with} \quad \begin{cases} P_0 = 1, & P_1 = \mu, & 1 - c = \gamma_a, \\ Q_1(-\lambda_k) = -\lambda_k Q_0(-\lambda_k) - 1. \end{cases} \end{aligned}$$

Hence, we obtain for the whole curly bracket in equation (15)

$$\begin{aligned} (A.1) \quad & 1 - 3\gamma_a \lambda_k \mu + \sum_{l=2}^L (2l+1) \{ \dots \} = \\ &= -a\lambda_k^2 + [1 + a\lambda_k^2 + \lambda_k(c + a\lambda_k^2) Q_0(-\lambda_k)] \sum_{l=0}^L (2l+1) P_l(-\lambda_k) P_l(\mu) - \\ & \quad - \lambda_k(c + a\lambda_k^2) \sum_{l=0}^L (2l+1) Q_l(-\lambda_k) P_l(\mu). \end{aligned}$$

Now we use the Christoffel formulae (see ERDÉLYI: *Higher Transcendental Functions*, p. 162, eqs. (20) and (31)) to calculate the two sums

$$\sum_{l=0}^L (2l+1) P_l(-\lambda_k) P_l(\mu) = \frac{L+1}{\mu + \lambda_k} [P_{L+1}(\mu) P_L(-\lambda_k) - P_L(\mu) P_{L+1}(-\lambda_k)],$$

and

$$\sum_{l=0}^L (2l+1)Q_l(-\lambda_k)P_l(\mu) = \frac{1}{\mu + \lambda_k} \{ (L+1)[P_{L+1}(\mu)Q_L(-\lambda_k) - P_L(\mu)Q_{L+1}(-\lambda_k)] - 1 \}.$$

Putting these values in equation (A.1) and replacing the Q 's with the aid of the W 's one gets

$$\begin{aligned} 1 - 3\gamma_a \lambda_k \mu + \sum_{l=2}^L (2l+1)\{\dots\} = & -a\lambda_k^2 + \frac{\lambda_k(c + a\lambda_k^2)}{\mu + \lambda_k} + \\ & + \frac{L+1}{\mu + \lambda_k} \{ [(1 + a\lambda_k^2)P_L(-\lambda_k) + \lambda_k(c + a\lambda_k^2)W_{L-1}(-\lambda_k)]P_{L+1}(\mu) - \\ & - [(1 + a\lambda_k^2)P_{L+1}(-\lambda_k) + \lambda_k(c + a\lambda_k^2)W_L(-\lambda_k)]P_L(\mu) \}. \end{aligned}$$

The coefficient of $P_L(\mu)$ in the last term vanishes because it is the characteristic equation (12) for $\lambda = -\lambda_k$. One may replace $1 + a\lambda_k^2$ in the coefficient of $P_{L+1}(\mu)$ by its value in the characteristic equation

$$1 + a\lambda_k^2 = -\lambda_k(c + a\lambda_k^2) \frac{W_L(-\lambda_k)}{P_{L+1}(-\lambda_k)},$$

and one obtains

$$\begin{aligned} 1 - 3\gamma_a \lambda_k \mu + \sum_{l=2}^L (2l+1)\{\dots\} = & -a\lambda_k^2 + \frac{\lambda_k(c + a\lambda_k^2)}{\mu + \lambda_k} - \\ & - \frac{\lambda_k(c + a\lambda_k^2)}{\mu + \lambda_k} \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} (L+1)[P_L(-\lambda_k)W_L(-\lambda_k) - P_{L+1}(-\lambda_k)W_{L-1}(-\lambda_k)]. \end{aligned}$$

The last bracket in this expression is a const. = $1/(L+1)$.

Hence we get finally for the expression

$$\begin{aligned} \text{(A.2)} \quad 1 - 3\gamma_a \lambda_k \mu + \sum_{l=2}^L (2l+1)[(1 + a\lambda_k^2)P_l(-\lambda_k) + \lambda_k(c + a\lambda_k^2)W_{l-1}(-\lambda_k)]P_l(\mu) = \\ = -a\lambda_k^2 + \frac{\lambda_k(c + a\lambda_k^2)}{\mu + \lambda_k} \left[1 - \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \right] \equiv \\ \equiv \frac{\lambda_k}{\mu + \lambda_k} \left\{ c - a\lambda_k \mu - (c + a\lambda_k^2) \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \right\}. \end{aligned}$$

For odd L is $P_{L+1}(-\lambda) = P_{L+1}(\lambda)$, therefore the choice of the sign of λ is arbitrary in $P_{L+1}(\lambda)$.

APPENDIX II

Calculation of the coefficients α_k .

In the following the solution of the equations (32)

$$\begin{aligned} \text{(32)} \quad f(0, -\mu_j) = \sum_{k=0,0,1,2,\dots}^{(L-1)/2} \alpha_k \lambda_k \left\{ -a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k - \mu_j} \right\} = 0 \quad \left(j = 1, 2, \dots, \frac{L+1}{2} \right) \\ \text{and} \quad \alpha_0 + \alpha_0 = 1, \end{aligned}$$

will be carried out in 2 ways, (a) a *straightforward way* and (b) a *shorter way*. Both ways show different interesting features.

(a) *The straightforward way.*

We take equation (32) for i and j and subtract $f(0, -\mu_i) - f(0, -\mu_j) = 0$ for two arbitrary $i, j = 1, 2, \dots (L+1)/2$, ($i \neq j$). Then we get rid of the term $-a\lambda_k$ in the curly bracket of (32):

$$(A.3) \quad \sum_k \alpha_k \lambda_k (c + a\lambda_k^2) \left[\frac{1}{\lambda_k - \mu_i} - \frac{1}{\lambda_k - \mu_j} \right] = \\ = (\mu_i - \mu_j) \sum_k \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{(\lambda_k - \mu_j)(\lambda_k - \mu_i)} = 0 \quad (i \neq j).$$

Originally we have $(L+3)/2$ unknown coefficients $\alpha_0, \alpha_1, \dots, \alpha_{(L-1)/2}$; this number is reduced by 1, however, in view of our condition of normalization $\alpha_0 + \alpha_0 = 1$. By the subtraction above one loses one linear independent equation, which may be represented by the sum of the eq. (32) over $j = 1, 2, \dots (L+1)/2$:

$$(A.4) \quad \sum_{j=1}^{(L+1)/2} f(0, -\mu_j) = \sum_k \sum_j \alpha_k \lambda_k \left\{ -a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k - \mu_j} \right\} = 0.$$

With the introduction of the polynomial

$$p(\lambda) = \prod_{j=1}^{(L+1)/2} (\lambda - \mu_j),$$

its logarithmic derivative

$$\frac{p'(\lambda)}{p(\lambda)} = \sum_{j=1}^{(L+1)/2} \frac{1}{\lambda - \mu_j},$$

and new variables b_k for equation (A.3)

$$(A.5) \quad \alpha_k \lambda_k (c + a\lambda_k^2) = b_k p(\lambda_k), \quad \left(k = \bar{0}, 0, 1, \dots \frac{L-1}{2} \right),$$

one may write both equations

$$(A.6) \quad \sum_k b_k \left[\prod_{l=1}^{(L+1)/2} (l \neq i, j) (\lambda_k - \mu_j) \right] = 0, \quad \left(i, j = 1, 2, \dots \frac{L+1}{2}; i \neq j \right),$$

and

$$(A.7) \quad \sum_k \alpha_k \lambda_k \left[-\frac{L+1}{2} a\lambda_k + (c + a\lambda_k^2) \frac{p'(\lambda_k)}{p(\lambda_k)} \right] = 0.$$

Let us consider first the set of $\frac{1}{2}((L+1)/2) \cdot ((L-1)/2)$ equations (A.6). They are equivalent to $(L-1)/2$ linear independent equations only; one recognizes this fact again by the following process. A power development of the product

$$\prod_{l=1}^{(L+1)/2} (l \neq i, j) (\lambda_k - \mu_l),$$

in powers of λ_k yields

$$(A.8) \quad \sum_k \{b_k \lambda_k^{(L-3)/2} - (\sum_{l \neq i, j}^{(l \neq i, j)} \mu_l) b_k \lambda_k^{(L-5)/2} + (\sum_{\substack{l, l' \\ l < l'}}^{(l, l' \neq i, j)} \mu_l \mu_{l'}) b_k \lambda_k^{(L-7)/2} - \\ - + - + \dots + (-1)^{(L-3)/2} (\prod_l^{(l \neq i, j)} \mu_l) b_k\} = 0.$$

A sufficient way (*) to satisfy these equations for every choice of i, j is to put to zero every term in the power development

$$(A.9) \quad \sum_k b_k \lambda_k^m = 0, \quad \left(m = 0, 1, 2, \dots \frac{L-3}{2}\right).$$

Then we have a simpler system of $(L-1)/2$ linear independent equations again for the $(L-1)/2$ unknown variables $b_1, b_2, \dots b_{(L-1)/2}$, if we regard the variables $b_{\bar{0}}$ and b_0 as known quantities just for a moment:

$$(A.10) \quad \sum_{s=1}^{(L-1)/2} b_s \lambda_s^m = -(b_{\bar{0}} \lambda_{\bar{0}}^m + b_0 \lambda_0^m). \quad \left(m = 0, 1, \dots \frac{L-3}{2}\right).$$

The solution of this system is with $r(\lambda) = \prod_{j=1}^{(L-1)/2} (\lambda - \lambda_j)$ and $r'(\lambda_s) = \prod_{j=1}^{(L-1)/2} (\lambda_s - \lambda_j)_{(j \neq s)}$:

$$(A.11) \quad b_s = \frac{1}{r'(\lambda_s)} \left\{ \frac{r(\lambda_{\bar{0}}) b_{\bar{0}}}{\lambda_s - \lambda_{\bar{0}}} + \frac{r(\lambda_0) b_0}{\lambda_s - \lambda_0} \right\}, \quad \left(s = 1, 2, \dots \frac{L-1}{2}\right),$$

or remembering $\lambda_{\bar{0}} = -\lambda_0$ and the relation (A.5) between the α'_s and b'_s

$$(A.12) \quad \alpha_s = \frac{\lambda_0(c + a\lambda_0^2)}{\lambda_s(c + a\lambda_s^2)} \frac{p(\lambda_s)}{r'(\lambda_s)} \left\{ \frac{\alpha_0 r(\lambda_0)}{(\lambda_s - \lambda_0)p(\lambda_0)} - \frac{\alpha_{\bar{0}} r(-\lambda_0)}{(\lambda_s + \lambda_0)p(-\lambda_0)} \right\}.$$

This equation relates the coefficients of the transients α_s to the coefficients $\alpha_{\bar{0}}$ and α_0 of the asymptotic part. Putting these values of α_s in equation (A.7) we obtain a relation between $\alpha_{\bar{0}}$ and α_0 ; this relation together with the normalization condition $\alpha_{\bar{0}} + \alpha_0 = 1$ determines $\alpha_{\bar{0}}$ and α_0 completely. To solve equation (A.7) we separate the two terms in the bracket and sum over

$$1) \quad -\frac{L+1}{2} a \sum_k \alpha_k \lambda_k^2 \quad \text{and} \quad 2) \quad \sum_k \alpha_k \lambda_k (c + a\lambda_k^2) \frac{p'(\lambda_k)}{p(\lambda_k)}$$

with the aid of the slightly extended Lagrange interpolation formula

$$(A.13) \quad f(\lambda) = R(\lambda) \left[\sum_j \frac{f(\lambda_j)}{(\lambda - \lambda_j) R'(\lambda_j)} + r_0 \right],$$

if $f(\lambda)$ and $R(\lambda)$ are polynomials of the same degree in λ . r_0 is a constant to be determined.

(*) See also: KY FAN and J. J. ANDREWS: *ORNL-Report 2485*.

(1) We consider first, using (A12)

$$\begin{aligned}
 \text{(A.14)} \quad a \sum_k \alpha_k \lambda_k^2 &= a(\alpha_{\bar{0}} + \alpha_0) \lambda_0^2 + a \sum_s \alpha_s \lambda_s^2 = a \lambda_0^2 + a \sum_s \alpha_s \lambda_s^2 = \\
 &= a \lambda_0^2 + \frac{\lambda_0(c + a \lambda_0^2)}{p(\lambda_0)p(-\lambda_0)} \sum_s \frac{[(\lambda_s + \lambda_0)\alpha_0 r(\lambda_0)p(-\lambda_0) - (\lambda_s - \lambda_0)\alpha_{\bar{0}} r(-\lambda_0)p(\lambda_0)] \lambda_s p(\lambda_s)}{(\lambda_s^2 - \lambda_0^2)(\lambda_s^2 + c/a)r'(\lambda_s)}.
 \end{aligned}$$

If one chooses

$$f(\lambda) = [(\lambda + \lambda_0)\alpha_0 r(\lambda_0)p(-\lambda_0) - (\lambda - \lambda_0)\alpha_{\bar{0}} r(-\lambda_0)p(\lambda_0)] \lambda p(\lambda)$$

and

$$R(\lambda) = (\lambda - \lambda_0)(\lambda^2 - \lambda_a^2)r(\lambda) \quad \text{with} \quad \lambda^2 - \lambda_a^2 = \lambda^2 + \frac{c}{a}, \quad \lambda_a = i \sqrt{\frac{c}{a}},$$

both polynomials $f(\lambda)$ and $R(\lambda)$ are of the same degree $(L+5)/2$ in λ . Therefore one has to find r_0 in the interpolation formula (A.13) by comparison of the coefficients of the largest power of λ ; it is

$$r_0 = \alpha_0 r(\lambda_0)p(-\lambda_0) - \alpha_{\bar{0}} r(-\lambda_0)p(\lambda_0).$$

Then a straightforward application of (A.13) at the points

$$\lambda_j = (-\lambda_a, \lambda_a, \lambda_0, \lambda_1, \dots, \lambda_{(L-1)/2}),$$

yields

$$\begin{aligned}
 \text{(A.15)} \quad a \sum_k \alpha_k \lambda_k^2 &= \frac{\lambda_0}{p(\lambda_0)p(-\lambda_0)} \left\{ (c + a \lambda_0^2)(\alpha_0 r(\lambda_0)p(-\lambda_0) - \alpha_{\bar{0}} r(-\lambda_0)p(\lambda_0)) + \right. \\
 &+ \frac{a}{2} [(\lambda_0 - \lambda_a)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 + \lambda_a)\alpha_{\bar{0}} r(-\lambda_0)p(\lambda_0)] \frac{p(-\lambda_a)}{r(-\lambda_a)} + \\
 &\left. + \frac{a}{2} [(\lambda_0 + \lambda_a)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 - \lambda_a)\alpha_{\bar{0}} r(-\lambda_0)p(\lambda_0)] \frac{p(\lambda_a)}{r(\lambda_a)} \right\}.
 \end{aligned}$$

(2) We consider secondly

$$\begin{aligned}
 \text{(A.16)} \quad \sum_k \alpha_k \lambda_k (c + a \lambda_k^2) \frac{p'(\lambda_k)}{p(\lambda_k)} &\equiv \lambda_0 (c + a \lambda_0^2) \left[\alpha_0 \frac{p'(\lambda_0)}{p(\lambda_0)} - \alpha_{\bar{0}} \frac{p'(-\lambda_0)}{p(-\lambda_0)} \right] + \\
 &+ \sum_s \alpha_s \lambda_s (c + a \lambda_s^2) \frac{p'(\lambda_s)}{p(\lambda_s)}.
 \end{aligned}$$

Putting α_s from equation (A.12) in the last sum one obtains

$$\begin{aligned}
 \text{(A.17)} \quad \sum_s \alpha_s \lambda_s (c + a \lambda_s^2) \frac{p'(\lambda_s)}{p(\lambda_s)} &= \\
 &= \frac{\lambda_0 (c + a \lambda_0^2)}{p(\lambda_0)p(-\lambda_0)} \sum_s \frac{[(\lambda_s + \lambda_0)\alpha_0 r(\lambda_0)p(-\lambda_0) - (\lambda_s - \lambda_0)\alpha_{\bar{0}} r(-\lambda_0)p(\lambda_0)] p'(\lambda_s)}{(\lambda_s^2 - \lambda_0^2)r'(\lambda_s)}.
 \end{aligned}$$

To this expression one has to apply the same kind of interpolation formula (A.13) with

$$f(\lambda) = [(\lambda + \lambda_0)\alpha_0 r(\lambda_0)p(-\lambda_0) - (\lambda - \lambda_0)\alpha_0 r(-\lambda_0)p(\lambda_0)]p'(\lambda),$$

and

$$R(\lambda) = (\lambda - \lambda_0)r(\lambda),$$

at the points $\lambda_j = (\lambda_0, \lambda_1, \dots, \lambda_{(L-1)/2})$. Both polynomials $f(\lambda)$ and $R(\lambda)$ are again of the same degree $(L+1)/2$ in λ . The coefficient of the largest power $\lambda^{(L-1)/2}$ of $p'(\lambda)$ is $(L+1)/2$; hence we find by comparison of the largest powers of $f(\lambda)$ and $R(\lambda)$

$$r_1 = \frac{L+1}{2} (\alpha_0 r(\lambda_0)p(-\lambda_0) - \alpha_0 r(-\lambda_0)p(\lambda_0)).$$

A straightforward application of the interpolation formula gives now

$$\begin{aligned} \text{(A.18)} \quad & \sum_s \frac{[(\lambda_s + \lambda_0)\alpha_0 r(\lambda_0)p(-\lambda_0) - (\lambda_s - \lambda_0)\alpha_0 r(-\lambda_0)p(\lambda_0)]p'(\lambda_s)}{(\lambda_s^2 - \lambda_0^2)r'(\lambda_s)} = \\ & = \frac{L+1}{2} (\alpha_0 r(\lambda_0)p(-\lambda_0) - \alpha_0 r(-\lambda_0)p(\lambda_0)) + \alpha_0 p(\lambda_0)p'(-\lambda_0) - \alpha_0 p(-\lambda_0)p'(\lambda_0). \end{aligned}$$

Putting the constituents together in \sum_k all terms with $p'(\lambda_0)$ and $p'(-\lambda_0)$ cancel; one obtains the simple result for the second sum

$$\begin{aligned} \text{(A.19)} \quad & \sum_k \alpha_k \lambda_k (c + a\lambda_k^2) \frac{p'(\lambda_k)}{p(\lambda_k)} = \\ & = \frac{L+1}{2} \frac{\lambda_0(c + a\lambda_0^2)}{p(\lambda_0)p(-\lambda_0)} [\alpha_0 r(\lambda_0)p(-\lambda_0) - \alpha_0 r(-\lambda_0)p(\lambda_0)]. \end{aligned}$$

Both sums (1) and (2) together yield now for (A.7)

$$\begin{aligned} \text{(A.20)} \quad & \sum_k \alpha_k \lambda_k \left[-\frac{L+1}{2} a\lambda_k + (c + a\lambda_k^2) \frac{p'(\lambda_k)}{p(\lambda_k)} \right] = -\frac{(L+1)a\lambda_0}{4p(\lambda_0)p(-\lambda_0)r(\lambda_a)r(-\lambda_a)} \cdot \\ & \cdot \{[(\lambda_0 - \lambda_a)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 + \lambda_a)\alpha_0 r(-\lambda_0)p(\lambda_0)]r(\lambda_a)p(-\lambda_a) + \\ & + [(\lambda_0 + \lambda_a)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 - \lambda_a)\alpha_0 r(-\lambda_0)p(\lambda_0)]r(-\lambda_a)p(\lambda_a)\} = 0. \end{aligned}$$

For $a \neq 0$ the curly bracket $\{\dots\} = 0$ gives now the required relation between α_0^- and α_0 . In the case of $\gamma_t = 1$, $a = 0$ we have to split a in two factors \sqrt{a} , because $\lim_{\gamma_t \rightarrow 1} \lambda_0 \sqrt{a} \rightarrow 1 + \frac{2}{5}\gamma_a + \dots$ remains finite and different from zero, and to multiply the curly bracket with a factor \sqrt{a} also for compensation of the denominator in $\lambda_a = i\sqrt{c/a}$. Correct in all cases is the quotient

$$\begin{aligned} \text{(A.21)} \quad & \frac{\alpha_0^-}{\alpha_0} = -\frac{r(\lambda_0)p(-\lambda_0)}{r(-\lambda_0)p(\lambda_0)} \frac{(\lambda_0 - \lambda_a)r(\lambda_a)p(-\lambda_a) + (\lambda_0 + \lambda_a)r(-\lambda_a)p(\lambda_a)}{(\lambda_0 + \lambda_a)r(\lambda_a)p(-\lambda_a) + (\lambda_0 - \lambda_a)r(-\lambda_a)p(\lambda_a)} = \\ & = \frac{r(\lambda_0)p(-\lambda_0)}{r(-\lambda_0)p(\lambda_0)} \frac{\lambda_a F_-(\lambda_a) - \lambda_0 F_+(\lambda_a)}{\lambda_a F_-(\lambda_a) + \lambda_0 F_+(\lambda_a)}, \end{aligned}$$

with the notation

$$(A.22) \quad F_{\pm}(\lambda) \equiv r(\lambda)p(-\lambda) \pm r(-\lambda)p(\lambda).$$

Finally we use the normalization condition $\alpha_0^- + \alpha_0 = 1$ or $1/\alpha_0 = 1 + \alpha_0^-/\alpha_0$ to obtain

$$(A.23) \quad \alpha_0 = \frac{\lambda_a F_-(\lambda_a) + \lambda_0 F_+(\lambda_a)}{\lambda_a F_+(\lambda_0) F_-(\lambda_a) - \lambda_0 F_-(\lambda_0) F_+(\lambda_a)} r(-\lambda_0)p(\lambda_0),$$

and

$$(A.24) \quad \alpha_0^- = \frac{\lambda_a F_-(\lambda_a) - \lambda_0 F_+(\lambda_a)}{\lambda_a F_+(\lambda_0) F_-(\lambda_a) - \lambda_0 F_-(\lambda_0) F_+(\lambda_a)} r(\lambda_0)p(-\lambda_0).$$

The introduction of the function

$$(34) \quad \delta(\lambda) = \lambda \frac{r(\lambda)p(-\lambda) + r(-\lambda)p(\lambda)}{r(\lambda)p(-\lambda) - r(-\lambda)p(\lambda)} \equiv \lambda \frac{F_+(\lambda)}{F_-(\lambda)},$$

has the advantage of a simple behaviour for $|\lambda| \rightarrow \infty$, as was shown in equation (60). It leads to a finite limit

$$\lim_{|\lambda| \rightarrow 0} \delta(\lambda) = M_1 - A_1 = \sum_{j=1}^{(L+1)/2} \mu_j - \sum_{s=1}^{(L-1)/2} \lambda_s,$$

or

$$\lim_{a \rightarrow 0} \delta(\lambda_a) = \lim_{a \rightarrow 0} \delta\left(i \sqrt{\frac{c}{a}}\right) = M_1 - A_1 \quad \text{also.}$$

Finally this way of calculation of the coefficients α_k shows by equation (A.9) that

$$\sum_k b_k \lambda_k^m = \sum_k \alpha_k \frac{c + a \lambda_k^2}{p(\lambda_k)} \lambda_k^{m+1} = 0 \quad \text{for } m = 0, 1, \dots, \frac{L-3}{2},$$

or for any arbitrary polynomial $f(\lambda)$ of degree $\leq (L-3)/2$ in λ

$$(A.25) \quad \sum_k \alpha_k \lambda_k \frac{(c + a \lambda_k^2) f(\lambda_k)}{p(\lambda_k)} = 0.$$

Examples for such $f(\lambda)$'s are $p''(\lambda)$, $r'(\lambda)$ etc.

A further result is contained in equation (A.15), if one considers that the sum of the 2nd and 3rd term of (A.15) vanishes in consequence of equation (A.20), namely the simple value for the sum

$$(A.26) \quad \sum_k \alpha_k \lambda_k^2 = \frac{\lambda_0(c + a \lambda_0^2)(\alpha_0 r(\lambda_0)p(-\lambda_0) - \alpha_0^- r(-\lambda_0)p(\lambda_0))}{ap(\lambda_0)p(-\lambda_0)} =$$

$$= -2N \lambda_0^2 \left(1 + \frac{a}{c} \lambda_0^2\right) \delta(\lambda_a),$$

after introduction of $\alpha_{\bar{0}}$ (equation (37)) and α_0 (equation (38)). $\sum_k \alpha_k \lambda_k^2$ is related to the second moment of the directed flux at the boundary.

(b) *The shorter way.*

A short calculation of the coefficients α_k uses the Lagrange interpolation formula as a generalization of a method, which was used first by WICK⁽⁸⁾. To this end we try to reconstruct formula (32) by a convenient interpolation, which allows one to read off the coefficients directly by comparison of the constructed equation with equation (32). To do that, we represent the polynomial $f(\lambda) = (C + D\lambda)p(\lambda)$ of the degree $(L+3)/2$ in λ twice — *first* by its values at the $(L+5)/2$ points $\lambda_j = (-\lambda_a, \lambda_a, \lambda_0, \lambda_1 \dots \lambda_{(L-1)/2})$ using the polynomial $R_1(\lambda) = (\lambda^2 - \lambda_a^2)(\lambda - \lambda_0)r(\lambda)$ of the degree $(L+5)/2$ in λ in the Lagrange interpolation formula

$$f(\lambda) = R(\lambda) \sum_j \frac{f(\lambda_j)}{(\lambda - \lambda_j)R'(\lambda_j)},$$

where λ_j runs over the series of points mentioned above and with $R(\lambda) = R_1(\lambda)$; *secondly* we repeat this process replacing the point λ_0 by $-\lambda_0$ without a change of the other points. This means, we use this time $R(\lambda) = R_2(\lambda) = (\lambda^2 - \lambda_a^2)(\lambda + \lambda_0)r(\lambda)$ in the Lagrange formula.

(1) $R(\lambda) = R_1(\lambda)$ yields (C, D are constants)

$$\begin{aligned} \text{(A.27)} \quad \frac{(C + D\lambda)p(\lambda)}{(\lambda^2 - \lambda_a^2)(\lambda - \lambda_0)r(\lambda)} = & \frac{(C - D\lambda_a)p(-\lambda_a)}{(\lambda + \lambda_a)2\lambda_a(\lambda_a + \lambda_0)r(-\lambda_a)} + \frac{(C + D\lambda_a)p(\lambda_a)}{(\lambda - \lambda_a)2\lambda_a(\lambda_a - \lambda_0)r(\lambda_a)} + \\ & + \frac{(C + D\lambda_0)p(\lambda_0)}{(\lambda - \lambda_0)(\lambda_0^2 - \lambda_a^2)r(\lambda_0)} + \sum_{s=1}^{(L-1)/2} \frac{(C + D\lambda_s)p(\lambda_s)}{(\lambda - \lambda_s)(\lambda_s^2 - \lambda_a^2)(\lambda_s - \lambda_0)r'(\lambda_s)}. \end{aligned}$$

(2) $R(\lambda) = R_2(\lambda)$ yields

$$\begin{aligned} \text{(A.28)} \quad \frac{(C + D\lambda)p(\lambda)}{(\lambda^2 - \lambda_a^2)(\lambda + \lambda_0)r(\lambda)} = & \frac{(C - D\lambda_a)p(-\lambda_a)}{(\lambda + \lambda_a)2\lambda_a(\lambda_a - \lambda_0)r(-\lambda_a)} + \frac{(C + D\lambda_a)p(\lambda_a)}{(\lambda - \lambda_a)2\lambda_a(\lambda_a + \lambda_0)r(\lambda_a)} + \\ & + \frac{(C - D\lambda_0)p(-\lambda_0)}{(\lambda + \lambda_0)(\lambda_0^2 - \lambda_a^2)r(-\lambda_0)} + \sum_{s=1}^{(L-1)/2} \frac{(C + D\lambda_s)p(\lambda_s)}{(\lambda - \lambda_s)(\lambda_s^2 - \lambda_a^2)(\lambda_s + \lambda_0)r'(\lambda_s)}. \end{aligned}$$

Both formulae differ only in the sign of λ_0 .

Now we choose the following special values in both equations: $\lambda = \mu_j$, for which $p(\lambda)$ is $p(\mu_j) = 0$ and the left sides of both equations vanish; further

⁽⁸⁾ G. C. WICK: *Zeits. f. Phys.*, **120**, 202 (1943).

we put $C = c$ and $D = a\mu_j$. Then we multiply the first equation by a constant A , the second by a constant B and add both together. We obtain

$$\begin{aligned}
 (A.29) \quad & B \frac{p(-\lambda_0)}{(\lambda_0^2 - \lambda_a^2)r(-\lambda_0)} \frac{c - a\lambda_0\mu_j}{\lambda_0 + \mu_j} - A \frac{p(\lambda_0)}{(\lambda_0^2 - \lambda_a^2)r(\lambda_0)} \frac{c + a\lambda_0\mu_j}{\lambda_0 - \mu_j} + \\
 & + \sum_{s=1}^{(L-1)/2} \left(\frac{-A}{\lambda_s - \lambda_0} + \frac{-B}{\lambda_s + \lambda_0} \right) \frac{p(\lambda_s)}{(\lambda_s^2 - \lambda_a^2)r(\lambda_s)} \frac{c + a\lambda_s\mu_j}{\lambda_s - \mu_j} = \\
 & = - \left\{ \frac{(c - a\lambda_0\mu_j)p(-\lambda_a)}{(\mu_j + \lambda_a)2\lambda_a r(-\lambda_a)} \left(\frac{A}{\lambda_a + \lambda_0} + \frac{B}{\lambda_a - \lambda_0} \right) + \right. \\
 & \quad \left. + \frac{(c + a\lambda_0\mu_j)p(\lambda_a)}{(\mu_j - \lambda_a)2\lambda_a r(\lambda_a)} \left(\frac{A}{\lambda_a - \lambda_0} + \frac{B}{\lambda_a + \lambda_0} \right) \right\}.
 \end{aligned}$$

We compare this equation with equation (32), which after separation of the terms with the indices $\bar{0}$ and 0 may be written in the following form

$$\begin{aligned}
 (32) \quad & \alpha_{\bar{0}}\lambda_0 \frac{c - a\lambda_0\mu_j}{\lambda_0 + \mu_j} + \alpha_0\lambda_0 \frac{c + a\lambda_0\mu_j}{\lambda_0 - \mu_j} + \sum_{s=1}^{(L-1)/2} \alpha_{\lambda_s} \frac{c + a\lambda_s\mu_j}{\lambda_s - \mu_j} = 0, \\
 & \left(j = 1, 2, \dots \frac{L+1}{2} \right),
 \end{aligned}$$

and we recognize that they would be comparable, if the right side of (A.29) could be put to zero. This, however, is possible, because the constants A and B are still arbitrary constants and because it turns out that A and B can be chosen independent of j . This will be shown now: the right side of (A.29) may be written

$$\begin{aligned}
 & - \frac{A}{2\lambda_a} \left\{ \frac{(c - a\lambda_0\mu_j)p(-\lambda_a)}{(\mu_j + \lambda_a)(\lambda_a + \lambda_0)r(-\lambda_a)} + \frac{(c + a\lambda_0\mu_j)p(\lambda_a)}{(\mu_j - \lambda_a)(\lambda_a - \lambda_0)r(\lambda_a)} \right\} - \\
 & - \frac{B}{2\lambda_a} \left\{ \frac{(c - a\lambda_0\mu_j)p(-\lambda_a)}{(\mu_j + \lambda_a)(\lambda_a - \lambda_0)r(-\lambda_a)} + \frac{(c + a\lambda_0\mu_j)p(\lambda_a)}{(\mu_j - \lambda_a)(\lambda_a + \lambda_0)r(\lambda_a)} \right\} = \\
 & = \frac{a}{2(\lambda_0^2 - \lambda_a^2)r(\lambda_a)r(-\lambda_a)} \{ A[(\lambda_0 - \lambda_a)r(\lambda_a)p(-\lambda_a) + (\lambda_0 + \lambda_a)r(-\lambda_a)p(\lambda_a)] - \\
 & \quad - B[(\lambda_0 + \lambda_a)r(\lambda_a)p(-\lambda_a) + (\lambda_0 - \lambda_a)r(-\lambda_a)p(\lambda_a)] \}.
 \end{aligned}$$

That μ_j is only contained in the curly brackets on the left side of the equation above as a factor $c + a\mu_j^2 = a(\mu_j^2 - \lambda_a^2)$ in this formula one owes to the fact that $c + a\lambda_a^2 = c + a(-c/a) = 0$, by reason of which the originally occurring terms $(c + a\lambda_a^2)\mu_j$ vanish. Hence A and B do not depend on the special μ_j chosen on the left side of (A.29), and the right side of equation (A.29) will vanish for

$$(A.30) \quad A = G[(\lambda_0 + \lambda_a)r(\lambda_a)p(-\lambda_a) + (\lambda_0 - \lambda_a)r(-\lambda_a)p(\lambda_a)],$$

$$(A.31) \quad B = G[(\lambda_0 - \lambda_a)r(\lambda_a)p(-\lambda_a) + (\lambda_0 + \lambda_a)r(-\lambda_a)p(\lambda_a)],$$

with an arbitrary constant G , which has to be determined later from our supposition of normalization $\alpha_0^- + \alpha_0 = 1$.

After that the equations (32) and (A.29) are in form suitable for comparison, which shows

$$(A.32) \quad \begin{cases} \alpha_0^- \lambda_0 = \frac{Bp(-\lambda_0)}{(\lambda_0^2 - \lambda_a^2)r(-\lambda_0)}, & \alpha_0 \lambda_0 = \frac{-Ap(\lambda_0)}{(\lambda_0^2 - \lambda_a^2)r(\lambda_0)}, \\ \alpha_s \lambda_s = - \left[\frac{A}{\lambda_s - \lambda_0} + \frac{B}{\lambda_s + \lambda_0} \right] \frac{p(\lambda_s)}{(\lambda_s^2 - \lambda_a^2)r'(\lambda_s)}. \end{cases}$$

One may use the first two equations to remove A and B from the last, whereupon one obtains a sometimes useful short representation of the coefficients α_s of the transients in terms of α_0^- and α_0 (again (A.12)).

$$(A.12) \quad \alpha_s = \frac{\lambda_0(c + a\lambda_0^2)}{\lambda_s(c + a\lambda_s^2)} \frac{p(\lambda_s)}{r'(\lambda_s)} \left\{ \frac{\alpha_0^- r(\lambda_0)}{(\lambda_s - \lambda_0)p(\lambda_0)} - \frac{\alpha_0 r(-\lambda_0)}{(\lambda_s + \lambda_0)p(-\lambda_0)} \right\},$$

$$\left(s = 1, 2, \dots, \frac{L-1}{2} \right).$$

From the normalization condition

$$(A.33) \quad \alpha_0^- + \alpha_0 = \frac{Br(\lambda_0)p(-\lambda_0) - Ar(-\lambda_0)p(\lambda_0)}{\lambda_0(\lambda_0^2 - \lambda_a^2)r(\lambda_0)r(-\lambda_0)} = 1,$$

one may find now the constant G , using the abbreviations (A.22)

$$(A.34) \quad G = \frac{\lambda_0(\lambda_0^2 - \lambda_a^2)r(\lambda_0)r(-\lambda_0)}{\lambda_0 F_-(\lambda_0) F_+(\lambda_a) - \lambda_a F_+(\lambda_0) F_-(\lambda_a)}.$$

We put this expression for G in the equations for the constants A , B

$$(A.35) \quad A = \frac{\lambda_0(\lambda_0^2 - \lambda_a^2)r(\lambda_0)r(-\lambda_0)[\lambda_0 F_+(\lambda_a) + \lambda_a F_-(\lambda_a)]}{\lambda_0 F_-(\lambda_0) F_+(\lambda_a) - \lambda_a F_+(\lambda_0) F_-(\lambda_a)},$$

$$(A.36) \quad B = \frac{\lambda_0(\lambda_0^2 - \lambda_a^2)r(\lambda_0)r(-\lambda_0)[\lambda_0 F_+(\lambda_a) - \lambda_a F_-(\lambda_a)]}{\lambda_0 F_-(\lambda_0) F_+(\lambda_a) - \lambda_a F_+(\lambda_0) F_-(\lambda_a)},$$

and in the equations for the coefficients α_0^- , α_0 , α_s

$$(A.37) \quad \begin{cases} \alpha_0^- = \frac{[\lambda_a F_-(\lambda_a) - \lambda_0 F_+(\lambda_a)]r(\lambda_0)p(-\lambda_0)}{\lambda_a F_+(\lambda_0) F_-(\lambda_a) - \lambda_0 F_-(\lambda_0) F_+(\lambda_a)}, \\ \alpha_0 = \frac{[\lambda_a F_-(\lambda_a) + \lambda_0 F_+(\lambda_a)]r(-\lambda_0)p(\lambda_0)}{\lambda_a F_+(\lambda_0) F_-(\lambda_a) - \lambda_0 F_-(\lambda_0) F_+(\lambda_a)}, \\ \alpha_s = \frac{[\lambda_a F_-(\lambda_a) + \lambda_s F_+(\lambda_a)]r(\lambda_0)r(-\lambda_0)}{\lambda_a F_+(\lambda_0) F_-(\lambda_a) - \lambda_0 F_-(\lambda_0) F_+(\lambda_a)} \frac{c + a\lambda_0^2}{c + a\lambda_s^2} \frac{2\lambda_0^2 p(\lambda_s)}{(\lambda_s^2 - \lambda_0^2)\lambda_s r'(\lambda_s)}, \end{cases}$$

and finally in the equation for the linear extrapolation distance

$$(A.38) \quad \delta = \frac{\lambda_0}{\alpha_0 - \alpha_0} = \lambda_0 \frac{\lambda_0 F_-(\lambda_0) F_+(\lambda_a) - \lambda_a F_+(\lambda_0) F_-(\lambda_a)}{\lambda_0 F_+(\lambda_0) F_+(\lambda_a) - \lambda_a F_-(\lambda_0) F_-(\lambda_a)}.$$

After that the solution is complete; the corresponding quantities are equal to those of the equations (37), (38), (39), if one remembers the abbreviations (34), (35), and (36).

APPENDIX III

Summation of the C-W part $f^I(0, \mu)$.

(1) The sum, which has to be carried out is

$$(17) \quad f^I(0, \mu) = \sum_{k=0,1,\dots}^{(L-1)/2} \alpha_k \lambda_k \left\{ -a \lambda_k + \frac{c + a \lambda_k^2}{\lambda_k + \mu} \right\}.$$

The first part of this sum was found already (see equation (A.26))

$$(A.39) \quad -a \sum_k \alpha_k \lambda_k^2 = \frac{\lambda_0(c + a \lambda_0^2)}{p(\lambda_0)p(-\lambda_0)} [\alpha_0 r(-\lambda_0) p(\lambda_0) - \alpha_0 r(\lambda_0) p(-\lambda_0)].$$

The second part is

$$(A.40) \quad \sum_k \alpha_k \lambda_k \frac{c + a \lambda_k^2}{\lambda_k + \mu} = \alpha_0 \lambda_0 \frac{c + a \lambda_0^2}{\lambda_0 + \mu} + \sum_{s=1}^{(L-1)/2} \alpha_s \lambda_s \frac{c + a \lambda_s^2}{\lambda_s + \mu}.$$

By introduction of the values of α_s from equation (A.12) in the last sum one obtains

$$\begin{aligned} \sum_{s=1}^{(L-1)/2} \alpha_s \lambda_s \frac{c + a \lambda_s^2}{\lambda_s + \mu} &= \\ &= \frac{\lambda_0(c + a \lambda_0^2)}{p(\lambda_0)p(-\lambda_0)} \sum_{s=1}^{(L-1)/2} \frac{[(\lambda_s + \lambda_0) \alpha_0 r(\lambda_0) p(-\lambda_0) - (\lambda_s - \lambda_0) \alpha_0 r(-\lambda_0) p(\lambda_0)] p(\lambda_s)}{(\lambda_s + \mu)(\lambda_s^2 - \lambda_0^2) r'(\lambda_s)}. \end{aligned}$$

Numerator and denominator in the sum are both of degree $(L+3)/2$ in λ_s . An application of the extended Lagrange interpolation formula (A.13) with

$$f(\lambda) = [(\lambda + \lambda_0) \alpha_0 r(\lambda_0) p(-\lambda_0) - (\lambda - \lambda_0) \alpha_0 r(-\lambda_0) p(\lambda_0)] p(\lambda),$$

and

$$R(\lambda) = (\lambda^2 - \lambda_0^2) r(\lambda),$$

at the $(L+3)/2$ points $\lambda_j = (-\lambda_0, \lambda_0, \lambda_1, \dots, \lambda_{(L-1)/2})$ requires a comparison of the largest powers $\lambda^{(L+3)/2}$ on both sides of the equation to find

$$r_0 = \alpha_0 r(\lambda_0) p(-\lambda_0) - \alpha_0^- r(-\lambda_0) p(\lambda_0).$$

Then a straightforward application of the interpolation formula with $\lambda = -\mu$ yields

$$\begin{aligned} \sum_s \frac{[(\lambda_s + \lambda_0) \alpha_0 r(\lambda_0) p(-\lambda_0) - (\lambda_s - \lambda_0) \alpha_0^- r(-\lambda_0) p(\lambda_0)] p(\lambda_s)}{(\lambda_s + \mu)(\lambda_s^2 - \lambda_0^2) r'(\lambda_s)} = \\ = \alpha_0 r(\lambda_0) p(-\lambda_0) - \alpha_0^- r(-\lambda_0) p(\lambda_0) + p(\lambda_0) p(-\lambda_0) \left(\frac{\alpha_0^-}{\mu - \lambda_0} - \frac{\alpha_0}{\mu + \lambda_0} \right) + \\ + \frac{[(\mu - \lambda_0) \alpha_0 r(\lambda_0) p(-\lambda_0) - (\mu + \lambda_0) \alpha_0^- r(-\lambda_0) p(\lambda_0)] p(-\mu)}{(\mu^2 - \lambda_0^2) r(-\mu)}. \end{aligned}$$

Hence, one obtains by addition of all parts of the sum (17) the Gauss quadrature part of the directed flux on the boundary:

$$(A.41) \quad f^i(0, \mu) = \frac{\lambda_0(c + a\lambda_0^2)}{p(\lambda_0)p(-\lambda_0)} \frac{(\lambda_0 + \mu)r(-\lambda_0)p(\lambda_0)\alpha_0^- + (\lambda_0 - \mu)r(\lambda_0)p(-\lambda_0)\alpha_0}{\lambda_0^2 - \mu^2} \frac{p(-\mu)}{r(-\mu)}.$$

Introduction of α_0^- and α_0 from equations (A.23, A.24) completes the expression

$$(A.42) \quad f^i(0, \mu) = \frac{2\lambda_0^2(c + a\lambda_0^2)r(\lambda_0)r(-\lambda_0)}{\lambda_a F_+(\lambda_0) F_-(\lambda_a) - \lambda_0 F_-(\lambda_0) F_+(\lambda_a)} \frac{\lambda_a F_-(\lambda_a) - \mu F_+(\lambda_a)p(-\mu)}{\lambda_0^2 - \mu^2} \frac{p(-\mu)}{r(-\mu)}.$$

Finally the use of $\delta(\lambda)$ equation (37), λ_a equation (38) and N equation (39) leads to the form of $f^i(0, \mu)$ in equation (41), because

$$\lambda_a F_-(\lambda_a) - \mu F_+(\lambda_a) = \lambda_a F_-(\lambda_a) \left[1 + \frac{a}{c} \mu \delta(\lambda_a) \right],$$

and N may be represented also in the following manner

$$(A.43) \quad N = \frac{r(\lambda_0)r(-\lambda_0)\lambda_a F_-(\lambda_a)}{\lambda_a F_+(\lambda_0) F_-(\lambda_a) - \lambda_0 F_-(\lambda_0) F_+(\lambda_a)}.$$

(2) A shorter way to find $f^i(0, \mu)$ is connected again with the superposition of A times equation (A.27) + B times equation (A.28) and the choice of $C = c$ and $D = -a\mu$ in the resulting expression. One gets by this procedure the formula

$$\begin{aligned} a \frac{p(-\mu)}{r(-\mu)} \left(\frac{-A}{\lambda_0 + \mu} + \frac{B}{\lambda_0 - \mu} \right) = -A \frac{(c - a\lambda_0\mu)p(\lambda_0)}{(\lambda_0 + \mu)(\lambda_0^2 - \lambda_a^2)r(\lambda_0)} + \\ + B \frac{(c + a\lambda_0\mu)p(-\lambda_0)}{(\lambda_0 - \mu)(\lambda_0^2 - \lambda_a^2)r(-\lambda_0)} + \sum_s \frac{(c - a\lambda_s\mu)p(\lambda_s)}{(\lambda_s + \mu)(\lambda_s^2 - \lambda_a^2)r'(\lambda_s)} \left(\frac{A}{\lambda_0 - \lambda_s} - \frac{B}{\lambda_0 + \lambda_s} \right), \end{aligned}$$

where A and B have the values of equations (A.30), (A.31). In consequence of (A.32)

$$\alpha_0 \lambda_0 = \frac{-Bp(-\lambda_0)}{(\lambda_0^2 - \lambda_a^2)r(-\lambda_0)}, \quad \alpha_0 \lambda_0 = \frac{-Ap(\lambda_0)}{(\lambda_0^2 - \lambda_a^2)r(\lambda_0)},$$

$$\alpha_s \lambda_s = \left[\frac{A}{\lambda_0 - \lambda_s} - \frac{B}{\lambda_0 + \lambda_s} \right] \frac{p(\lambda_s)}{(\lambda_s^2 - \lambda_a^2)r'(\lambda_s)},$$

we have on the right side just

$$\sum_{k=0,1,\dots}^{(L-1)/2} \alpha_k \lambda_k = \frac{c - a\lambda_k \mu}{\lambda_k + \mu},$$

which is $f'(0, \mu)$. Hence, again

$$(A.42) \quad f'(0, \mu) = \left[\frac{aB}{\lambda_0 - \mu} - \frac{aA}{\lambda_0 + \mu} \right] \frac{p(-\mu)}{r(-\mu)} =$$

$$= \frac{2\lambda_0^2(c + a\lambda_0^2)r(\lambda_0)r(-\lambda_0)}{\lambda_0 F_-(\lambda_0) F_+(\lambda_a) - \lambda_a F_+(\lambda_0) F_-(\lambda_a)} \frac{F_+(\lambda_a)\mu - \lambda_a F_-(\lambda_a)}{\lambda_0^2 - \mu^2} \frac{p(-\mu)}{r(-\mu)}.$$

APPENDIX IV

Calculation of the current at the boundary.

To calculate

$$(A.44) \quad \sum_{k=0,1,\dots}^{(L-1)/2} \alpha_k \lambda_k \equiv -\lambda_0 \alpha_0^- + \lambda_0 \alpha_0 + \sum_{s=1}^{(L-1)/2} \alpha_s \lambda_s \equiv -(\alpha_0^- - \alpha_0) \lambda_0 + \sum_s \alpha_s \lambda_s,$$

we use equation (A.12), which gives the α_s 's in terms of α_0^- and α_0 :

Then we have

$$(A.45) \quad \sum_s \alpha_s \lambda_s =$$

$$= \frac{\lambda_0(c + a\lambda_0^2)}{ap(\lambda_0)p(-\lambda_0)} \sum_s \frac{[(\lambda_0 + \lambda_s)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 - \lambda_s)\alpha_0^- r(-\lambda_0)p(\lambda_0)]p(\lambda_s)}{(\lambda_s^2 - \lambda_0^2)(\lambda_s^2 - \lambda_a^2)r'(\lambda_s)}.$$

We apply the Lagrange interpolation formula in the simple form (44) to a function

$$f(\lambda) = [(\lambda + \lambda_0)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 - \lambda)r(-\lambda_0)p(\lambda_0)]p(\lambda) \text{ of the degree } \frac{L+3}{2},$$

in λ with an aid function $R(\lambda) = (\lambda - \lambda_0)(\lambda^2 - \lambda_a^2)r(\lambda)$ of degree $(L+5)/2$ in λ at its $(L+5)/2$ zeros $\lambda_j = (-\lambda_a, \lambda_a, \lambda_0, \lambda_1, \dots, \lambda_{(L-1)/2})$. Then we obtain

$$\begin{aligned} & \frac{[(\lambda_0 + \lambda)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 - \lambda)\alpha_0 r(-\lambda_0)p(\lambda_0)]p(\lambda)}{(\lambda - \lambda_0)(\lambda^2 - \lambda_a^2)r(\lambda)} = \\ &= \frac{[(\lambda_0 - \lambda_a)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 + \lambda_a)\alpha_0 r(-\lambda_0)p(\lambda_0)]p(-\lambda_a)}{(\lambda + \lambda_a)2\lambda_a(\lambda_0 + \lambda_a)r(-\lambda_a)} - \\ & - \frac{[(\lambda_0 + \lambda_a)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 - \lambda_a)\alpha_0 r(-\lambda_0)p(\lambda_0)]p(\lambda_a)}{(\lambda - \lambda_a)2\lambda_a(\lambda_0 - \lambda_a)r(\lambda_a)} + \\ & + \frac{2\lambda_0\alpha_0 p(\lambda_0)p(-\lambda_0)}{(\lambda - \lambda_0)(\lambda_0^2 - \lambda_a^2)} + \sum_s \frac{[(\lambda_0 + \lambda_s)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 - \lambda_s)\alpha_0 r(-\lambda_0)p(\lambda_0)]p(\lambda_s)}{(\lambda - \lambda_s)(\lambda_s^2 - \lambda_0^2)r'(\lambda_s)}. \end{aligned}$$

Putting $\lambda = -\lambda_0$ we find the sum \sum_s , required above:

$$\begin{aligned} (A.46) \quad & \sum_s \frac{[(\lambda_0 + \lambda_s)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 - \lambda_s)\alpha_0 r(-\lambda_0)p(\lambda_0)]p(\lambda_s)}{(\lambda_s^2 - \lambda_0^2)(\lambda_s^2 - \lambda_a^2)r'(\lambda_s)} = \\ &= \frac{p(\lambda_0)p(-\lambda_0)}{\lambda_0^2 - \lambda_a^2} (\alpha_0 - \alpha_0) + \frac{[(\lambda_0 + \lambda_a)\alpha_0 r(\lambda_0)p(-\lambda_0) + (\lambda_0 - \lambda_a)\alpha_0 r(-\lambda_0)p(\lambda_0)]p(\lambda_a)}{\lambda_a(\lambda_0^2 - \lambda_a^2)r(\lambda_a)}. \end{aligned}$$

The last term on the right side is found by an application of the relation (A.21) between α_0^- and α_0 .

Putting (A.45) with (A.46) in equation (A.44), the first two terms cancel. We obtain for the sum considered in (A.44)

$$\sum_{k=0,1,\dots}^{(L-1)/2} \alpha_k \lambda_k = \frac{\lambda_0}{\lambda_a} \frac{(\lambda_0 + \lambda_a)r(\lambda_0)p(-\lambda_0)\alpha_0 + (\lambda_0 - \lambda_a)r(-\lambda_0)p(\lambda_0)\alpha_0^-}{p(\lambda_0)p(-\lambda_0)} \frac{p(\lambda_a)}{r(\lambda_a)}.$$

After introducing α_0 and α_0^- (see equations (A.23), (A.24)) we get

$$\begin{aligned} (A.47) \quad & \sum_{k=0,1,\dots}^{(L-1)/2} \alpha_k \lambda_k = \frac{2\lambda_0^2 r(\lambda_0)r(-\lambda_0)p(\lambda_a)[F_+(\lambda_a) + F_-(\lambda_a)]}{r(\lambda_a)[\lambda_a F_+(\lambda_0) F_-(\lambda_a) - \lambda_0 F_-(\lambda_0) F_+(\lambda_a)]} = \\ &= \frac{2\lambda_0^2 N p(\lambda_a)}{r(\lambda_a)} \frac{F_+(\lambda_a) + F_-(\lambda_a)}{\lambda_a F_-(\lambda_a)} \quad [\text{with } N \text{ of eq. (A.43)}] = \\ &= \frac{4N\lambda_0^2 p(\lambda_a)p(-\lambda_a)}{\lambda_a[r(\lambda_a)p(-\lambda_a) - r(-\lambda_a)p(\lambda_a)]}. \end{aligned}$$

In the last equation the abbreviations $F_{\pm}(\lambda)$ used in Appendix II were replaced by their definitions, see equation (A.22). The last form of equation (A.47) was applied to the evaluation of the formula (55) for the current at the boundary $z = 0$.

II. - Behavior of the P_L -solution of the Milne problem with anisotropic scattering for $L \rightarrow \infty$. (Algebraic part).

In Part I it was shown that the partial solutions of the Chandrasekhar-Wick method which contain the factor $g(\mu)$ cf. Eq. (I-27) ^(*), dependent on the directions $\mu = \cos \vartheta$, are solutions of the *non-truncated Boltzmann integro-differential equation* (I-1). An error occurs only because the Gauss quadrature method is used to evaluate approximately the integrals on the right side of (I-1). The integrands are not polynomials but are meromorphic functions of μ ; hence, the integral is evaluated only approximately. This error is manifested in L -approximation as an error in the characteristic equation, which is an algebraic equation instead of a transcendental. But the sequence of algebraic characteristic equations converges for $L \rightarrow \infty$ to the exact transcendental characteristic equation. Therefore one may expect that the solutions of the C-W method tend to the correct Wiener-Hopf solution for $L \rightarrow \infty$.

The solutions (I-16) of the spherical harmonics method, however, can be used for a correct integration of the integrals on the right side of Eq. (I-1). They are continuous functions in the whole interval $-1 \leq \mu \leq 1$ and their μ -dependence is of such a kind to satisfy a truncated system of differential equations (compare Sect. 1 in I) instead of equation (I-1). The approximate characteristic equation of the S-H method coincides with that of the C-W method, but the *truncation of the infinite system of differential equations* (I-3) introduces a further, and more serious, error. It is shown in the following Sect. 1 that the spherical harmonics method introduces an « *error-source* » in the Boltzmann equation. It turns out that this error-source is important especially at those places where the contribution of the transients to the solution is large. In the Milne problem this region is near the boundary of the medium. The error-source does not show a tendency to vanish in the limit $L \rightarrow \infty$ in this case, indicating a lack of convergence of the spherical harmonics method. In a denumerably infinite set of directions, however, the spherical harmonics solution coincides with the C-W solution in the limit $L \rightarrow \infty$ and the error-source is zero there also. Hence it is possible to devise a scheme for passing to the limit by using only the C-W part of the S-H solution.

(*) The numeral I in the number of an equation refers to the corresponding equation in the first part of the paper.

Although the convergence of the angular distribution $f(\zeta, \mu)$ given by the spherical harmonics method presents serious difficulties, it will be seen that the convergence of the total flux $f_{00}(\zeta)$ shows no such difficulties⁽¹⁰⁾. Thus for applications to reactors—for example, in the calculation of thermal utilization—where the spherical harmonics method is used really to compute $f_{00}(\zeta)$, not $f(\zeta, \mu)$, subtle questions of convergence do not arise: the spherical harmonics method, used in the usual straightforward way, does converge to the proper answer in the limit $L \rightarrow \infty$. The performance of this limit-process, however, will be published in a later paper; it is partially carried out in the author's ORNL-report No 2358.

1. — The error-source in the Boltzmann equation caused by spherical harmonics solutions.

If one puts the solution $f(\zeta, \mu)$ (I-16) of the spherical harmonics method in P_L -approximation in the Boltzmann integro-differential equation (I-1), one obtains a remainder

$$(1) \quad R(\zeta, \mu) = \mu \frac{\partial f(\zeta, \mu)}{\partial \zeta} + f(\zeta, \mu) - \frac{c}{2} \int_{-1}^{+1} f(\zeta, \mu') d\mu' - \frac{1}{2} \frac{a}{1-c} \mu \int_{-1}^{+1} \mu' f(\zeta, \mu') d\mu' = \\ = \sum_{k=0,1,\dots}^{(L-1)/2} \alpha_k \exp[\zeta/\lambda_k] \left\{ (\mu + \lambda_k) \left[-a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k + \mu} \left(1 - \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \right) \right] - c + a\mu\lambda_k \right\},$$

using the integrals (I-51, 54). In (1) all terms which do not contain spherical harmonics cancel one another. Hence, the remainder is

$$(2) \quad R(\zeta, \mu) = - \sum_k \alpha_k \exp[\zeta/\lambda_k] (c + a\lambda_k^2) \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} = \\ = - \frac{P_{L+1}(\mu)}{P_{L+1}(0)} \sum_k \alpha_k \exp[\zeta/\lambda_k] (c + a\lambda_k^2) \frac{P_{L+1}(0)}{P_{L+1}(-\lambda_k)}, \\ (3) \quad = \frac{P_{L+1}(\mu)}{P_{L+1}(0)} f^u(z, 0).$$

The last equation was found by comparison of the \sum_k with equation (I-18)

⁽¹⁰⁾ For further information and illustration see the section « Convergence of the Spherical Harmonics Method » in Chapter IX of A. M. WEINBERG and E. P. WIGNER: *Theory of Neutron Chain Reactions, Preliminary Draft*, 12-9-57-Revision, Oak Ridge Nat. Lab., Oak Ridge, Tenn., USA.

for $\mu = 0$. Therefore, the remainder is closely related to $f^{II}(z, 0)$ i.e., to the difference between the spherical harmonics and Gauss quadrature method solutions for $\mu = +0$. Instead of fulfilling the Boltzmann equation (I-1) exactly, $f(z, \mu) = f^I(z, \mu) + f^{II}(z, \mu)$ satisfies the equation

$$(4) \quad \mu \frac{\partial f(\zeta, \mu)}{\partial \zeta} + f(\zeta, \mu) = \\ = \frac{1}{2} (1 - \gamma_a) \int_{-1}^{+1} f(\zeta, \mu') d\mu' + \frac{3}{2} (1 - \gamma_i) \mu \int_{-1}^{+1} f(\zeta, \mu') \mu' d\mu' + \frac{P_{L+1}(\mu)}{P_{L+1}(0)} f^{II}(z, 0),$$

with a « 2^{L+1} -pole-source», the strength of which depends on the part $f^{II}(z, 0)$ of the directed flux perpendicular to the z -axis in every z -plane.

This result is valid also for every partial solution contained in equation (I-16) with a corresponding partial remainder contained in equation (4). Therefore it is not a special feature of the Milne problem, but a common feature of all spherical harmonics solutions. The remainders of the different partial solutions, however, may accumulate or destroy one another. One recognizes that the true Boltzmann equation can be satisfied for arbitrary μ ($-1 \leq \mu \leq 1$) only for problems, in which the boundary conditions are of such a kind that $f^{II}(z, 0)$ tends to zero in the limit $L \rightarrow \infty$ as $1/L^\alpha$ with $\alpha > \frac{1}{2}$.

This will *presumably* not happen in the Milne problem, because in this case $f^{II}(z, 0)$ has just the task of joining smoothly the directed fluxes of medium and vacuum side at the point of discontinuity $z = 0$; hence one cannot expect that $\lim_{L \rightarrow \infty} f^{II}(z, 0) = 0$. On the boundary between medium and vacuum we expect $\lim_{L \rightarrow \infty} f^{II}(0, 0) = -\frac{1}{2} \lim_{L \rightarrow \infty} f^I(0, 0)$, if the spherical harmonics series tends to $\frac{1}{2}$ of the discontinuity of the exact Wiener-Hopf solution. Normalized to unit density at the boundary the exact values are $f(0, \mu = +0) = \frac{1}{2}$, $f(0, \mu = -0) = 0$ with a discontinuity of $\frac{1}{2}$ in the case of isotropic scattering without absorption. The spherical harmonics method yields with the same normalization in $L = 3$ approximation $f(0, 0) = 0.3737$, in $L = 5$ approximation $f(0, 0) = 0.3382$, in $L = 15$ approximation (*) $f(0, 0) = 0.2898$. This behavior does not contradict to the expectation above, according to which one expects $\lim_{L \rightarrow \infty} f(0, 0) = \frac{1}{4}$ or $\lim_{L \rightarrow \infty} f^{II}(0, 0) = -\frac{1}{4}$.

For the special directions $\mu = \mu_j$ ($j = 1, 2, \dots, (L+1)/2$) the error in (4) vanishes, because $P_{L+1}(\mu_j) = 0$. From equations (I.16), (I.17), (I.18) we know that in these directions $f^{II}(z, \mu_j) = 0$ and $f(z, \mu_j) = f^I(z, \mu_j)$ which is identical with the Gauss quadrature solution for $\mu = +\mu_j$ and is zero for $\mu = -\mu_j$ (this last was our boundary condition for the Milne problem in the spherical harmonics method). Therefore we may fulfil the boundary condition and the

(*) I owe the calculation of the P_{15} -approximation to Mr. H. S. MORAN, Oak Ridge National Laboratory.

true integro-differential equation in the limit $L \rightarrow \infty$ at a denumerable infinite set of directions $\mu = \pm \mu_j$. In these directions the spherical harmonics- and Gauss quadrature-method solutions coincide, in all other intermediate directions they differ and the spherical harmonics method leads in general to a non-vanishing remainder in (4). Therefore we have to select in the limit $L \rightarrow \infty$ only the denumerably infinite set of directions $\mu = \mu_j$, in which both methods coincide, if we wish to proceed to the exact limiting function of the Wiener-Hopf method.

The following consideration shows the behavior of the remainder in equation (4) as function of μ at a fixed z . If $f^{\text{II}}(z, 0)$ is finite and different from zero, we obtain the quotient of the remainders at the end-points of the μ -interval $|\mu| = 1$ and the center $\mu = 0$, ($L+1$ even),

$$\frac{R(\zeta, 1)}{R(\zeta, 0)} = \frac{P_{L+1}(1)}{P_{L+1}(0)} = (-1)^{(L+1)/2} \frac{2 \cdot 4 \cdot 6 \cdot 8 \dots (L+1)}{1 \cdot 3 \cdot 5 \cdot 7 \dots L}.$$

This quotient is 2.67 for $L=3$, -3.2 for $L=5$, 5.09 for $L=15$. For large L it diverges as \sqrt{L}

(5)
$$\frac{R(\zeta, 1)}{R(\zeta, 0)} \sim (-1)^{(L+1)/2} \sqrt{\frac{\pi}{2}} \sqrt{L+1}.$$

Hence the remainder in (4) increases strongly from the center to the end-points of the μ -interval.

The divergence proportional to \sqrt{L} , however, occurs only in the narrowest neighborhood of the point $\mu=1$. One recognizes this fact if one considers the quotient of the remainder at the extrema μ^E of $P_{L+1}(\mu)$ and the remainder at the center $\mu=0$. The abscissae μ^E of the extrema, for which $P'_{L+1}(\mu^E) = 0$ are approximatively $\mu^E_m \sim \cos((m+\frac{1}{4})/(L+\frac{3}{2}))\pi$ for large L , where the numeration $m=1, 2, \dots, (L+1)/2$ begins with $m=1$ for the largest abscissa between the zeros μ_1 and μ_2 of $P_{L+1}(\mu_j) = 0$ [$\mu_2 < \mu^E_1 < \mu_1$] and ends with $\mu^E_{(L+1)/2} = \cos \pi/2$ for $m = (L+1)/2$. The extrema are

$$P_{L+1}(\mu^E_m) \sim (-1)^m \sqrt{\frac{2}{\pi(L+1)}} [1 - (\mu^E_m)^2]^{-\frac{1}{4}},$$

for large L . Hence the quotient of the remainders at fixed z for large L is

(6)
$$\frac{R(\zeta, \mu^E_m)}{R(\zeta, 0)} \sim (-1)^{(L+1)/2-m} [1 - (\mu^E_m)^2]^{-\frac{1}{4}}.$$

The absolute value of this quotient is independent of L and increases slowly with the distance from the center. The divergence proportional to \sqrt{L} occurs only in the small interval between the largest zero of $P_{L+1}(\mu)$,

$\mu_1 \approx \cos [(3\pi/2)/(2L+3)] \approx 1 - \frac{1}{2}((3\pi/2)/(2L+3))^2 < \mu \leq 1$; this interval in the neighborhood of $\mu=1$ decreases as L^{-2} for large L .

Hence the spherical harmonics approximation shows two possibilities for a lack of convergence. (1) If $f^{II}(z, 0)$ does not vanish in the limit $L \rightarrow \infty$, an event, which may be caused by the physical conditions of the problem; (2) If $f^{II}(z, 0)$ does not vanish strongly enough to reduce the remainder to zero also for $|\mu|=1$; this is the case, if $f^{II}(z, 0)$ does not tend to zero stronger than $1/\sqrt{L}$ for $L \rightarrow \infty$.

In our case of the Milne problem one observes a quick decrease of the remainder (3) from the boundary to the interior of the medium $\zeta \leq 0$. To recognize this fact we separate the $\alpha_{\bar{v}}$ and α_0 -terms from the sum in the remainder ($L+1$ even):

$$(7) \quad R(\zeta, \mu) = - \left\{ (c + a\lambda_0^2) \frac{P_{L+1}(\mu)}{P_{L+1}(\lambda_0)} (\alpha_{\bar{v}} \exp[-\zeta/\lambda_0] + \alpha_0 \exp[\zeta/\lambda_0]) + \sum_{s=1}^{(L-1)/2} \alpha_s \exp[\zeta/\lambda_s] (c + a\lambda_s^2) \frac{P_{L+1}(\mu)}{P_{L+1}(\lambda_s)} \right\}.$$

λ_0 lies mostly outside of the real interval $(-1, 1)$ and is usually large; in this region we have for large L

$$(8) \quad P_{L+1}(\lambda_0) \sim \frac{[\lambda_0 + (\lambda_0^2 - 1)^{\frac{1}{2}}]^{L+\frac{1}{2}}}{\sqrt{2\pi(L+1)(\lambda_0^2 - 1)^{\frac{1}{2}}}}.$$

μ however lies in the interior of the real interval $(-1, 1)$ and here is

$$(9) \quad P_{L+1}(\mu) \sim \sqrt{\frac{2}{\pi(L+1)\sqrt{1-\mu^2}}} \sin \varphi \quad \text{with} \quad \varphi = \frac{\pi}{4} + \left(L + \frac{3}{2}\right) \arccos \mu,$$

for large L and $1 - \varepsilon < \mu < 1 + \varepsilon$ (ε small). Therefore the quotient

$$\frac{P_{L+1}(\mu)}{P_{L+1}(\lambda_0)} \sim \frac{\sqrt{2} \sin \varphi}{(1 - \mu^2)^{\frac{1}{2}} (2\lambda_0)^{L+1}},$$

tends to zero for $L \rightarrow \infty$ in this μ -interval. The last statement remains valid at $\mu=1$ because $P_{L+1}(\pm 1) = 1$ ($L+1$ even). Hence the first term in (7) plays no role in the limit $L \rightarrow \infty$; *the remainder contains in the limit $L \rightarrow \infty$ only the contributions of the transients (with $0 < \lambda_s < 1$)*

$$(10) \quad \lim_{L \rightarrow \infty} R(\zeta, \mu) = \lim_{L \rightarrow \infty} \left\{ - \sum_{s=1}^{(L-1)/2} \alpha_s \exp[\zeta/\lambda_s] (c + a\lambda_s^2) \frac{P_{L+1}(\mu)}{P_{L+1}(\lambda_s)} \right\}.$$

The meaning of this fact is that the spherical harmonics method produces a remainder *in the angular distribution* just in those points, in which one wants better information than diffusion theory can give.

Introduction of the values (I-39) of α_s in (9) leads to

$$(11) \quad \lim_{L \rightarrow \infty} R(\zeta, \mu) = \lim_{L \rightarrow \infty} \left\{ N(c + a\lambda_0^2) 2\lambda_0^2 p(\mu) p(-\mu) \sum_{s=1}^{(L-1)/2} \frac{\exp[\zeta/\lambda_s] [1 - (a/c)\lambda_s \delta(\lambda_s)]}{\lambda_s(\lambda_0^2 - \lambda_s^2) r'(\lambda_s) p(-\lambda_s)} \right\}$$

Because the sign of $r'(\lambda_s)$ alternates with s , the sum in (11) is the difference of two infinite series with alternating terms. This renders a general statement of the behavior of (11) difficult.

One may use equation (4) to form the moments $\int_{-1}^{+1} \mu^l f(\zeta, \mu) d\mu$. Because the remainder is proportional to $P_{L+1}(\mu)$ and $\int_{-1}^{+1} \mu^l P_{L+1}(\mu) d\mu = 0$ for $0 \leq l \leq L$ the relations between the moments of $f(\zeta, \mu)$, which one could derive on the basis of the correct integro-differential equation (I-1), are the same in the spherical harmonics P_L -approximation up to $l = L$. One of those relations is equation (I-25), it is the case $l = 0$. In general, if we denote the moments by $m_l(\zeta) = \int_{-1}^{+1} \mu^l f(\zeta, \mu) d\mu$ the correct integral equation (I-1) and equation (4) for P_L -approximation in spherical harmonics method give equally the following relations connecting the moments

$$\frac{\partial}{\partial \zeta} m_{l+1}(\zeta) + m_l(\zeta) = \begin{cases} \frac{1 - \gamma_a}{l + 1} m_0(\zeta) & \text{for } l \text{ even} \\ 3 \frac{1 - \gamma_t}{l + 2} m_1(\zeta) & \text{for } l \text{ odd} \end{cases}$$

for linearly anisotropic scattering and $0 \leq l \leq L$.

2. - The functions, which tend for positive and negative μ in the limit $L \rightarrow \infty$ to the exact directed flux.

We put in this section and index L on the approximate P_L -spherical harmonics solutions for the directed flux $f_L(\zeta, \mu) = f_L^I(\zeta, \mu) + f_L^{II}(\zeta, \mu)$ (I-16) which satisfies the Boltzmann equation (4) with the error source term

$$(13) \quad \mu \frac{\partial f_L(\zeta, \mu)}{\partial \zeta} + f_L(\zeta, \mu) = F_L(\zeta, \mu),$$

where

$$(14) \quad F_L(\zeta, \mu) = \frac{1}{2} (1 - \gamma_a) \int_{-1}^{+1} f_L(\zeta, \mu') d\mu' + \frac{3}{2} (1 - \gamma_t) \mu \int_{-1}^{+1} \mu' f_L(\zeta, \mu') d\mu' + \frac{P_{L+1}(\mu)}{P_{L+1}(0)} f_L^I(\zeta, 0).$$

The exact directed flux, which satisfies the Boltzmann equation (I-1), may be denoted in this section by $f(\zeta, \mu)$ without index:

$$(15) \quad \mu \frac{\partial f(\zeta, \mu)}{\partial \zeta} + f(\zeta, \mu) = F(\zeta, \mu),$$

where

$$(16) \quad F(\zeta, \mu) = \frac{1}{2} (1 - \gamma_a) \int_{-1}^{+1} f(\zeta, \mu') d\mu' + \frac{3}{2} (1 - \gamma_e) \mu \int_{-1}^{+1} \mu' f(\zeta, \mu') d\mu'.$$

If one considers μ for the moment as a fixed parameter and solves equations (13) and (15) as inhomogeneous linear differential equations, one obtains

$$(17) \quad f(\zeta, \mu) = \exp[-\zeta/\mu] \left\{ C(\mu) + \frac{1}{\mu} \int_0^{\zeta} F(\zeta', \mu) \exp[\zeta'/\mu] d\zeta' \right\},$$

for functions with or without index L , differing only by different « constants » $C(\mu)$. The constants $C(\mu)$ are determined by the boundary conditions. For large negative ζ we have the asymptotic behavior $f(\zeta, \mu) \sim \exp[-\zeta/\lambda_{0\infty}]$ and $f_L(\zeta, \mu) \sim \exp[-\zeta/\lambda_0]$ respectively, with $\lambda_{0\infty}$ and $\lambda_0 > 1$ for positive μ ; hence one may use, for positive μ , the boundary conditions $\exp[\zeta/\mu] f(\zeta, \mu) \rightarrow 0$ and $\exp[\zeta/\mu] f_L(\zeta, \mu) \rightarrow 0$ for $\zeta \rightarrow -\infty$; these conditions determine the constants $C(\mu)$ and $C_L(\mu)$, because $\mu \leq 1$. This yields in the first case (exact solution)

$$\lim_{\zeta \rightarrow -\infty} \exp[\zeta/\mu] f(\zeta, \mu) = C(\mu) + \frac{1}{\mu} \int_0^{-\infty} F(\zeta', \mu) \exp[\zeta'/\mu] d\zeta' = 0,$$

or

$$(18) \quad C(\mu) = \frac{1}{\mu} \int_{-\infty}^0 F(\zeta', \mu) \exp[\zeta'/\mu] d\zeta' \quad (0 < \mu \leq 1),$$

and

$$(19) \quad f(\zeta, \mu) = \frac{\exp[-\zeta/\mu]}{\mu} \int_{-\infty}^{\zeta} F(\zeta', \mu) \exp[\zeta'/\mu] d\zeta'.$$

In the second case (P_L -approximation) the source boundary condition leads to

$$(20) \quad C_L(\mu) = \frac{1}{\mu} \int_{-\infty}^0 F_L(\zeta', \mu) \exp[\zeta'/\mu] d\zeta' \quad (0 < \mu \leq 1),$$

and

$$(21) \quad f_L(\zeta, \mu) = \frac{\exp[-\zeta/\mu]}{\mu} \int_{-\infty}^{\zeta} F_L(\zeta', \mu) \exp[\zeta'/\mu] d\zeta'.$$

If we take the difference of the exact and the approximate directed fluxes in the limit $L \rightarrow \infty$ we obtain for positive μ

$$(22) \quad f(\zeta, \mu) - \lim_{L \rightarrow \infty} f_L(\zeta, \mu) = \frac{1}{\mu} \exp[-\zeta/\mu] \lim_{L \rightarrow \infty} \int_{-\infty}^{\zeta} [F(\zeta', \mu) - F_L(\zeta', \mu)] \exp[\zeta'/\mu] d\zeta'.$$

We now assume that the approximate density and current tend to the exact value for $L \rightarrow \infty$; hence the corresponding terms cancel in the integrand on the right side of (22) and the error source term in $F_L(\zeta, \mu)$ —compare equations (14) and (16)—alone remains:

$$\begin{aligned} (23) \quad f(\zeta, \mu) - \lim_{L \rightarrow \infty} f_L(\zeta, \mu) &= -\frac{1}{\mu} \exp[-\zeta/\mu] \lim_{L \rightarrow \infty} \frac{P_{L+1}(\mu)}{P_{L+1}(0)} \int_{-\infty}^{\zeta} f_L^{\text{II}}(\zeta', 0) \exp[\zeta'/\mu] d\zeta' = \\ &= \frac{1}{\mu} \exp[-\zeta/\mu] \lim_{L \rightarrow \infty} \sum_{k=0,1,\dots}^{(L-1)/2} \alpha_k (c + a\lambda_k^2) \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \int_{-\infty}^{\zeta} \exp\left[\zeta' \left(\frac{1}{\mu} + \frac{1}{\lambda_k}\right)\right] d\zeta' = \\ &= \exp[-\zeta/\mu] \lim_{L \rightarrow \infty} \sum_k \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{\mu + \lambda_k} \exp\left[\zeta \left(\frac{1}{\mu} + \frac{1}{\lambda_k}\right)\right] \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} = \\ &= \lim_{L \rightarrow \infty} \sum_k \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{\mu + \lambda_k} \exp[\zeta/\lambda_k] \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} = -\lim_{L \rightarrow \infty} f_L^{\text{II}}(\zeta, \mu). \quad (0 < \mu \leq 1). \end{aligned}$$

Evidently one may include the direction $\mu = 0$ for $\zeta \leq 0$ in formula (23). Therefore one finds again that the Gauss quadrature part $f_L^{\text{I}}(\zeta, \mu)$ tends for positive μ (forward scattering) to the exact directed flux

$$(24) \quad f(\zeta, \mu) = \lim_{L \rightarrow \infty} [f_L(\zeta, \mu) - f_L^{\text{II}}(\zeta, \mu)] = \lim_{L \rightarrow \infty} f_L^{\text{I}}(\zeta, \mu), \quad (0 \leq \mu \leq 1).$$

For negative μ (backward scattering) we have different boundary conditions for the exact and the approximate functions, which determine the constants $C(\mu)$ and $C_L(\mu)$. In the first case we may use the condition of no flux from the vacuum to the medium at the boundary $\zeta = 0$ i.e. $f(0, \mu) = 0$ for negative μ . Equation (17) yields in this case

$$(25) \quad f(0, \mu) = C(\mu) = 0 \quad (-1 \leq \mu < 0)$$

and we have simply

$$(26) \quad f(\zeta, \mu) = \frac{\exp[-\zeta/\mu]}{\mu} \int_0^{\zeta} F(\zeta', \mu) \exp[\zeta'/\mu] d\zeta'. \quad (-1 \leq \mu < 0).$$

For the approximate directed flux, however, this boundary condition holds only for the discrete set of points $\mu = -\mu_j$ ($j = 1, 2, \dots, (L+1)/2$), where $f_L(0, -\mu_j) = 0$. In general we have a directed flux $f_L(0, \mu)$ backward from the vacuum to the medium at the boundary $\zeta = 0$. Hence equation (17) gives

$$(27) \quad C_L(\mu) = f_L(0, \mu) \quad (-1 \leq \mu < 0)$$

and the directed flux in L -approximation is

$$(28) \quad f_L(\zeta, \mu) = \exp[-\zeta/\mu] \left[f_L(0, \mu) + \frac{1}{\mu} \int_0^{\zeta} F_L(\zeta', \mu) \exp[\zeta'/\mu] d\zeta' \right]. \quad (-1 \leq \mu < 0).$$

Taking again the difference between the exact and the approximate directed fluxes in the limit $L \rightarrow \infty$ we find for negative μ

$$(29) \quad f(\zeta, \mu) - \lim_{L \rightarrow \infty} f_L(\zeta, \mu) = \\ = -\exp[-\zeta/\mu] \lim_{L \rightarrow \infty} \left\{ f_L(0, \mu) + \frac{1}{\mu} \int_0^{\zeta} [F_L(\zeta', \mu) - F(\zeta', \mu)] \exp[\zeta'/\mu] d\zeta' \right\} = \\ = -\exp[-\zeta/\mu] \lim_{L \rightarrow \infty} \left\{ f_L(0, \mu) + \frac{1}{\mu} \frac{P_{L+1}(\mu)}{P_{L+1}(0)} \int_0^{\zeta} f_L^u(\zeta', 0) \exp[\zeta'/\mu] d\zeta' \right\}.$$

The last integral is

$$\frac{P_{L+1}(\mu)}{P_{L+1}(0)} \int_0^{\zeta} f_L^u(\zeta', 0) \exp[\zeta'/\mu] d\zeta' = -\sum_k \alpha_k (c + a\lambda_k^2) \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \int_0^{\zeta} \exp \left[\zeta' \left(\frac{1}{\mu} + \frac{1}{\lambda_k} \right) \right] d\zeta' = \\ = -\mu \sum_k \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{\mu + \lambda_k} \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \left(\exp \left[\zeta \left(\frac{1}{\mu} + \frac{1}{\lambda_k} \right) \right] - 1 \right).$$

By introducing this in equation (29) we get

$$f(\zeta, \mu) - \lim_{L \rightarrow \infty} f_L(\zeta, \mu) = \\ = \lim_{L \rightarrow \infty} \left\{ -f_L(0, \mu) \exp[-\zeta/\mu] + \sum_k \frac{\alpha_k \lambda_k (c + a\lambda_k^2)}{\mu + \lambda_k} \frac{P_{L+1}(\mu)}{P_{L+1}(-\lambda_k)} \left[\exp[\zeta/\lambda_k] - \exp[-\zeta/\mu] \right] \right\} = \\ = \lim_{L \rightarrow \infty} \left\{ -f_L(0, \mu) \exp[-\zeta/\mu] - f_L^u(\zeta, \mu) + f_L^u(0, \mu) \exp[-\zeta/\mu] \right\} = \\ = -\lim_{L \rightarrow \infty} \{ f_L^u(0, \mu) \exp[-\zeta/\mu] + f_L^u(\zeta, \mu) \}. \quad (-1 \leq \mu < 0),$$

remembering equation (I-18) and $f_L(0, \mu) = f_L^I(0, \mu) + f_L^{II}(0, \mu)$. Hence, we see that the exact directed flux $f(\zeta, \mu)$ will be, in the limit $L \rightarrow \infty$, the sum of two terms, both containing only the first part f_L^I (the Gauss quadrature part) of the spherical harmonics solution f_L

$$(30) \quad f(\zeta, \mu) = \lim_{L \rightarrow \infty} [f_L^I(\zeta, \mu) - f_L^I(0, \mu) \exp[-\zeta/\mu]] \quad (-1 \leq \mu < 0).$$

The second term removes the poles of $f_L^I(\zeta, \mu)$ at $\mu = -\lambda_k$; this can be seen quickly by remembering the expression (I-17) of $f_L^I(\zeta, \mu)$. We therefore have the following representation of equation (30) for negative μ

$$(31) \quad f(\zeta, \mu) = \lim_{L \rightarrow \infty} \sum_{k=0,1,\dots}^{(L-1)/2} \alpha_k \lambda_k \left\{ -a\lambda_k + \frac{c + a\lambda_k^2}{\lambda_k + \mu} \right\} (\exp[\zeta/\lambda_k] - \exp[-\zeta/\mu]).$$

In equation (30) both ζ and μ are negative, their quotient ζ/μ is always positive. In the interior of the medium—excluding the boundary $\zeta = 0$ —the term $\exp[-\zeta/\mu]$ vanishes with $\mu \rightarrow 0$. Hence, we have a continuous transition to the expression (24) for positive μ . On the boundary $\zeta = 0$ however, we have to exclude the direction $\mu = 0$ perpendicular to the z -axis from equation (31) and we obtain $f(0, \mu) = 0$ for all $\mu < 0$. In this way the well-known discontinuity arises between the expressions (31) and (24) for $\mu = 0$ on the boundary of the medium $\zeta = 0$.

* * *

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Sur une généralisation non linéaire de la mécanique ondulatoire et les propriétés des fonctions d'ondes correspondantes.

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SOMMAIRE. — 1. Introduction. — 2. Les solutions principales de l'équation de Klein-Gordon. — 3. Les ondes planes généralisées déduites des fonctions elliptiques de Jacobi. — 4. Les solutions ondes planes des équations d'ondes non linéaires $\square \psi + \alpha \psi + \gamma \psi^3 = 0$. — 5. La composition des fonctions d'ondes dans les théories non linéaires. — 6. Solutions invariantes et solutions radiales des équations précédentes.

1. — Introduction.

Je me propose d'exposer ici quelques résultats que j'ai obtenu dans la recherche et l'étude des solutions de quelques types d'équations d'ondes non linéaires susceptibles de généraliser les équations d'ondes de la mécanique ondulatoire.

De nombreux auteurs ont déjà cherché à introduire des équations d'ondes non linéaires en partant soit d'une étude phénoménologique des interactions, soit en cherchant une théorie non linéaire dont la théorie quantique des champs soit une approximation.

J'ai adopté un tout autre point de vue en cherchant si des considérations très générales pouvaient sinon indiquer exactement, tout au moins restreindre les classes d'équations d'ondes non linéaires susceptibles d'être introduites. Partant d'une analyse des types de solutions de l'équation d'ondes de Klein-Gordon, j'ai été amené [16-18] à en examiner les généralisations acceptables. Réciproquement, ces généralisations satisfont à des équations d'ondes

généralisations de l'équation de Klein-Gordon. Dans le cas des ondes planes j'ai été amené [16] ainsi à retrouver un type d'équations non linéaires déjà rencontré par R. FINKELSTEIN, R. LE LEVIER, M. RUDERMAN [6], L. SCHIFF [20] et N. ROSEN et H. B. ROSENSTOCK [19].

2. — Les solutions principales de l'équation de Klein-Gordon.

La mécanique ondulatoire usuelle représente les corpuscules sans spin par des fonctions d'ondes $\psi(x, y, z, t)$ solutions de l'équation de Klein-Gordon

$$(1) \quad \begin{cases} \square \psi + \mu_0^2 \psi = 0, \\ \square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right), \quad \mu_0 = \frac{m_0 c}{\hbar}, \quad \hbar = \frac{h}{2\pi}. \end{cases}$$

Bien que des théorèmes généraux montrent l'équivalence de tous les systèmes complets de solutions des équations d'ondes, les applications de la mécanique ondulatoire montrent qu'il est nécessaire suivant les problèmes examinés, d'utiliser des systèmes de base adaptés possédant par exemple des symétries particulières. Dans une généralisation non linéaire de la mécanique ondulatoire qui semble souhaitable à différents égards il est possible que certains systèmes de base soient seuls à considérer, l'équivalence entre système pouvant ne résulter que d'une dégénérescence associée à l'approximation linéaire.

Suivant les problèmes étudiés les principaux types de solutions de l'équation de Klein-Gordon sont:

- a) les solutions du type ondes planes,
- b) » » ondes invariantes,
- c) » » ondes sphériques,
- d) » » ondes guidées.

a) Les solutions du type « ondes planes » s'obtiennent à partir de l'équation (1) en supposant que les fonctions $\psi(x, y, z, t)$ ne dépendent que d'une seule variable soit τ , combinaison linéaire de x, y, z, t

$$\tau = \frac{1}{\hbar} [Wt - (\mathbf{p}\mathbf{x})] = Kct - (\mathbf{K}\mathbf{x}),$$

par l'intermédiaire de quatre constantes (W, P_1, P_2, P_3) ou (K, K_1, K_2, K_3) telles que

$$W^2 = c^2 p^2 + m_0^2 c^4 \quad \text{ou} \quad K^2 = \mathbf{K}^2 + \mu_0^2;$$

La fonction $\psi(\tau)$ solution de (1) est alors solution de l'équation différentielle

$$(2) \quad \frac{d^2\psi(\tau)}{d\tau^2} + \psi(\tau) = 0.$$

La solution générale de (2) est une combinaison de deux types de solutions, les unes paires ψ_c , les autres impaires ψ_s .

$$(3) \quad \psi_c = A \cos \tau, \quad \psi_s = B \sin \tau.$$

Les fonctions $\psi(\tau) = \psi(Kct - (\mathbf{K}\mathbf{x}))$ peuvent être considérées comme résultant d'une transformation de Lorentz appliquée à la solution particulière du système propre $\psi(t)$, solution de (1) indépendante de x, y, z . On a alors

$$\tau = \mu_0 ct = \frac{2\pi}{h} m_0 c^2 t.$$

Les ondes planes (3) soient $\psi(\tau)$ forment un système complet de solutions de (1), fonctions de τ , uniformes, périodiques et d'amplitudes bornées.

b) Les solutions invariantes s'obtiennent à partir de (1) en considérant les solutions particulières de cette équation qui ne dépendent que d'une seule variable celle-ci étant un invariant relativiste.

On prend généralement pour cette variable

$$(4) \quad u = \sqrt{c^2 t^2 - r^2},$$

ou

$$u^2 = c^2 t^2 - (x^2 + y^2 + z^2).$$

On voit facilement que

$$(5) \quad \square = \frac{d^2}{du^2} + \frac{3}{u} \frac{d}{du}.$$

L'équation (1) détermine alors $\psi(u)$ par

$$(6) \quad \left[\frac{d^2}{du^2} + \frac{3}{u} \frac{d}{du} + \mu_0^2 \right] \psi(u) = 0.$$

C'est encore une équation différentielle dont la solution générale s'exprime au moyen de fonctions de Bessel d'ordre un

$$(7) \quad \psi(u) = \frac{A}{u} J_1(\mu_0 u) + \frac{B}{u} N_1(\mu_0 u) = \frac{C_1}{u} H_1^{(1)}(\mu_0 u) + \frac{C_2}{u} H_1^{(2)}(\mu_0 u),$$

(J_1 , N_1 fonctions de Bessel de première et de seconde espèces, $H_1^{(1)}$ et $H_1^{(2)}$ fonctions de Hankel d'ordre 1 correspondantes).

c) et d) Pour l'introduction des ondes sphériques et des ondes guidées [18] nous allons maintenant supposer qu'il existe un repère privilégié R_0 dans lequel les fonctions d'ondes $\psi(x, y, z, t)$ s'expriment sous forme du produit d'une fonction de t , $\psi_1(t)$ et d'une fonction des variables d'espace, $\psi_2(x, y, z)$ ou $\psi_2(r, \theta, \varphi)$:

$$(8) \quad \psi(\mathbf{x}, t) = \psi_1(t)\psi_2(x, y, z) = \psi_1(t)\psi_2(r, \theta, \varphi).$$

Nous aurons alors

$$\psi_2(\mathbf{x}) \frac{1}{c^2} \frac{d^2 \psi_1}{dt^2}(t) - \psi_1(t) \Delta \psi_2(\mathbf{x}) + \mu_0^2 \psi_1 \psi_2 = 0.$$

Introduisant deux constantes, λ_1 , λ_2 telles que

$$\lambda_1 - \lambda_2 = \mu_0^2,$$

$\psi_1(t)$ et $\psi_2(\mathbf{x})$ satisferont aux équations

$$(9) \quad \begin{cases} \frac{1}{c^2} \frac{d^2 \psi_1}{dt^2}(t) + \lambda_1 \psi_1(t) = 0, \\ \Delta \psi_2(x, y, z) + \lambda_2 \psi_2(x, y, z) = 0. \end{cases}$$

(Nous supposerons λ_1 et λ_2 réels et nous nous bornerons ici au cas $\lambda_1 > 0$ afin de ne pas introduire de solutions de type évanescents par rapport à t . Ces solutions amorties au cours du temps ne doivent pas être écartées dans une étude générale que je ne fais pas ici).

Nous obtenons alors pour la fonction $\psi_1(t)$

$$(10) \quad \psi_1(t) = c_1 \exp[i\sqrt{\lambda_1}ct] + c_2 \exp[-i\sqrt{\lambda_1}ct] = c'_1 \cos(\sqrt{\lambda_1}ct) + c'_2 \sin(\sqrt{\lambda_1}ct).$$

Pour $\psi_2(x, y, z)$ deux cas sont à considérer:

$$1) \lambda_1 > \mu_0^2, \lambda_2 > 0.$$

$$\Delta \psi_2 + \lambda_2 \psi_2 = 0,$$

admet alors pour solutions acceptables

$$(11) \quad \psi_2(r, \theta, \varphi) = \frac{1}{\sqrt{r}} [AJ_{l+\frac{1}{2}}(\sqrt{\lambda_2}r) + RN_{l+\frac{1}{2}}(\sqrt{\lambda_2}r)] y_l^m(\theta, \varphi);$$

$$2) \lambda_1 < \mu_0^2, \lambda_2 < 0.$$

$$\Delta \psi_2 - |\lambda_2| \psi_2 = 0,$$

a pour solutions restant bornées quand $r \rightarrow \infty$

$$(12) \quad \psi_2(r, \theta, \varphi) = \frac{A}{\sqrt{r}} K_{l+\frac{1}{2}}(\sqrt{|\lambda_2|}r) y_l^m(\theta, \varphi).$$

Si nous nous bornons au cas $l = 0$ nous n'avons plus à considérer les fonctions sphériques $y_l^m(\theta, \varphi)$ et il reste

$$(13) \quad \psi_2(r, \theta, \varphi) = \psi_2(r).$$

On a donc dans les cas ci-dessus

$$(14) \quad \psi_2(r) = A' \frac{\sin(\sqrt{\lambda_2}r)}{r} + B' \frac{\cos(\sqrt{\lambda_2}r)}{r}.$$

$$(15) \quad \psi_2(r) = \frac{A''}{r} \exp[-\sqrt{|\lambda_2|}r].$$

Les solutions dites « ondes sphériques » de la mécanique ondulatoire orthodoxe sont obtenues à partir de ces expressions en posant

$$(16) \quad \lambda_1 = \frac{W^2}{\hbar^2 c^2} = K^2, \quad \lambda_2 = \frac{p^2}{\hbar^2} = |\mathbf{K}|^2,$$

$$\text{d'où} \quad \lambda_1 - \lambda_2 = K^2 - |\mathbf{K}|^2 = \mu_0^2.$$

On a alors nécessairement $\lambda_1 > \mu_0^2$ et pour $l = 0$ la solution onde sphérique générale s'écrit

$$(17) \quad \psi_{\text{sph.}} = \psi_1(t)\psi_2(r) = [c'_1 \cos Kct + c'_2 \sin Kct] A' \left[\frac{\sin |\mathbf{K}|r}{r} + A'' \frac{\cos |\mathbf{K}|r}{r} \right] = \\ = c''_1 \frac{\sin(Kct \mp |\mathbf{K}|r)}{r} + c''_2 \frac{\cos(Kct \mp |\mathbf{K}|r)}{r}.$$

La mécanique ondulatoire orthodoxe considère également le cas particulier des solutions ci-dessus pour lequel on a

$$(18) \quad \lambda_1 = 0, \quad \psi = \psi_2(r).$$

On est alors dans le cas 2) ci-dessus. $\lambda_1 = 0$ entraîne $|\lambda_2| = \mu_0^2$ et

$$(19) \quad \psi = \psi(r) = \psi_2(r) = \frac{c_0}{r} \exp[-\mu_0 r].$$

Cette solution, en fixant la valeur de la constante C_0 , est considérée comme représentant le champ $\psi(r)$ créé par une source C_0 localisée au point $r = 0$ dans le système propre du corpuscule (ici le repère R_0).

On passe des solutions $\psi = \psi_1(t)\psi_2(x, y, z)$ avec $\psi_1(t)$, $\psi_2(x, y, z)$ donnés par (10), (11) et (12) ou (10), (14) et (15) aux solutions du type « Ondes guidées » pour lesquelles le corpuscule est localisé et décrit une trajectoire (rectiligne et uniforme en l'absence de champ extérieur) en effectuant sur les fonctions ψ une transformation de Lorentz dépendant explicitement du temps.

Pour un corpuscule se déplaçant le long de l'axe OZ avec la vitesse v nous poserons

$$ct = \cosh \gamma ct' - \sinh \gamma z', \quad z = \cosh \gamma z' - \sinh \gamma ct',$$

$$x = x', \quad y = y', \quad \tanh \gamma = v,$$

d'où

$$r^2 = x'^2 + y'^2 + \cosh^2 \gamma (z' - \tanh \gamma ct')^2,$$

$$\sqrt{\lambda_1} t = \sqrt{\lambda_1} (\cosh \gamma ct' - \sinh \gamma z').$$

Ecrivant encore

$$\sqrt{\lambda_1} = \mu_1, \quad K_1 = \mu_1 \cosh \gamma, \quad \mathbf{K}_1 = \mu_1 \sinh \gamma,$$

$$\sqrt{\lambda_1} t = K_1 ct' - |\mathbf{K}_1| z',$$

$$r^2 = x'^2 + y'^2 + \left(\frac{K_1}{\mu_1} \right)^2 [z' - vt']^2 = \varrho'^2,$$

nous obtenons alors l'expression de l'« onde guidée »

$$(20) \quad \psi'(x', y', z', t') =$$

$$= \left\{ \frac{c'_1 \cos}{c'_1 \sin} (K_1 ct' - |\mathbf{K}_1| z') \right\} \cdot \left[A' \frac{\sin(\sqrt{\lambda_2} \varrho')}{\varrho'} + A'' \frac{\cos(\sqrt{\lambda_2} \varrho')}{\varrho'} \right].$$

Le terme en A'' introduit une singularité polaire qui se déplace avec la vitesse v ($\varrho' = 0$ pour $x' = y' = 0$, $z' = vt'$). Même si l'on se borne à la partie régulière, une structure définie dans R_0 par une combinaison de solutions du type (10), (11) et (12) ou (10), (14) et (15) engendre une solution combinaison des $\psi'(x', y', z', t')$ ci-dessus déplacée avec la vitesse v .

La solution particulière

$$\psi = \psi(r) = \frac{c_0 \exp[-\mu_0 r]}{r},$$

conduit à la solution guidée

$$(21) \quad \psi(x', y', z', t') = \frac{c_0 \exp[-\mu_0 (x'^2 + y'^2 + \cosh^2 \gamma (z' - vt')^2)]}{[x'^2 + y'^2 + \cosh^2 \gamma (z' - vt')^2]^{\frac{1}{2}}},$$

interprétée ordinairement comme champ de Yukawa d'une source en mouvement rectiligne et uniforme de vitesse $\text{tgh } \gamma = v$.

De même les solutions du type onde plane rentrent dans le schéma des ondes guidées lorsque nous posons $\lambda_2 = 0$, $\lambda_1 = 0$, $\psi = \psi_1(t)$.

3. — Les ondes planes généralisées déduites des fonctions elliptiques de Jacobi.

Dans une extension de la mécanique ondulatoire basée sur l'équation de Klein-Gordon nous devons généraliser soit l'ensemble des types de solutions que nous venons de considérer soit seulement certaines d'entre elles que des raisons physiques nous conduisent à considérer comme rattachées plus directement à la représentation de la matière.

Si nous considérons d'abord les solutions du type ondes planes, nous avons vu qu'elles pouvaient être considérées comme résultant d'une transformation du groupe de Lorentz appliquée aux solutions particulières du système propre

$$(22) \quad \psi_a = A' \sin \tau_0, \quad \psi_c = A'' \cos \tau_0,$$

avec

$$(23) \quad \tau_0 = \mu_0 ct = \frac{2\pi}{h} m_0 c^2 t = 2\pi \nu_0 t.$$

Cette forme de solution met en évidence un caractère fondamental de la représentation des corpuscules en mécanique ondulatoire sur lequel M. L. DE BROGLIE a souvent insisté: Dans le système propre du corpuscule la fonction d'ondes associe à celui-ci une « horloge », c'est-à-dire une fonction périodique du temps propre de période $T = h/m_0 c^2$ (ou de fréquence $\nu_0 = m_0 c^2/h$).

Si nous voulons généraliser cette conception tout en essayant d'enrichir la notion de corpuscule en introduisant non plus la seule constante intrinsèque $\nu_0 = m_0 c^2/h$ mais deux ou plusieurs constantes, la généralisation la plus immédiate consiste à prendre comme fonction d'ondes représentant le corpuscule dans son système propre, au lieu des fonctions circulaires $\cos \tau$ ou $\sin \tau$, certaines des fonctions elliptiques de Jacobi possédant une période réelle et une période imaginaire pure. La définition de ces fonctions introduit un paramètre k réel, compris entre 0 et 1. Pour $k = 0$, ces fonctions redonnent $\sin \tau$ et $\cos \tau$. La donnée de k équivaut à introduire un paramètre intrinsèque supplémentaire.

La théorie des fonctions de Jacobi introduit trois fonctions principales:

$\text{sn}(u, k)$	de périodes	$4K$ et $4iK'$,
$\text{cn}(u, k)$	»	$4K$ et $4iK'$,
$\text{dn}(u, k)$	»	$2K$ et $2iK'$.

Les périodes $K(k)$ et $K'(k)$ sont définies par l'intégrale

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

et par

$$K'(k) = K(k'), \quad \text{avec } k'^2 = 1 - k^2.$$

A partir de ces trois fonctions on construit un système de 12 fonctions elliptiques en adjoignant à $\text{sn } u$, $\text{cn } u$, $\text{dn } u$ leurs inverses et leurs quotients:

$$\begin{aligned} \text{ns } u &= \frac{1}{\text{sn } u}, & \text{nc } u &= \frac{1}{\text{cn } u}, & \text{nd } u &= \frac{1}{\text{dn } u}, \\ \text{sc } u &= \text{tn } u = \frac{\text{sn } u}{\text{cn } u}, & \text{sd } u &= \frac{\text{sn } u}{\text{dn } u}, \\ \text{cs } u &= \frac{\text{cn } u}{\text{sn } u}, & \text{cd } u &= \frac{\text{cn } u}{\text{dn } u}, \\ \text{ds } u &= \frac{\text{dn } u}{\text{sn } u}, & \text{dc } u &= \frac{\text{dn } u}{\text{cn } u}, \end{aligned}$$

on a notamment entre ces fonctions les relations

$$\begin{aligned} \text{sn}(u + K, k) &= \text{cd}(u, k), \\ \text{cn}(u + K, k) &= -k' \text{sd}(u, k), \\ \text{dn}(u + K, k) &= k' \text{nd}(u, k), \\ \text{sn}(u, 0) &= \sin u, & \text{sn}(u, 1) &= \text{tgh } u, \\ \text{cn}(u, 0) &= \cos u, & \text{cn}(u, 1) &= \frac{1}{\cosh u}, \\ \text{dn}(u, 0) &= 1, & \text{dn}(u, 1) &= \frac{1}{\cosh u}. \end{aligned}$$

On trouvera dans de nombreux livres de mathématiques appliquées l'étude des propriétés de ces fonctions. A titre indicatif je ne citerai que les ouvrages d'APPEL et LACOUR [1], de GREENHILL [9], de TRICOMI [22] et l'excellente petite monographie de BOWMAN [2].

La généralisation des fonctions d'ondes solutions ondes planes de l'équation de Klein-Gordon nous amène à poser $u = \tau$ d'où

$$(24) \quad \tau = 4K(k)v_0 t = 4K(k) \frac{m_0 c^2}{\hbar} t = \frac{m_0 c^2}{\hbar'} t = \mu_0 c t.$$

$4K$ est ici l'analogie du facteur 2π du cas trigonométrique et ceci nous conduit à introduire une nouvelle constante de Planck réduite

$$(25) \quad \hbar' = \frac{h}{4K(k)},$$

remplaçant la constante usuelle $\hbar = h/2\pi$, μ_0 sera lié à la masse dynamique m_0 par l'intermédiaire de \hbar' ,

$$(26) \quad \mu_0 = \frac{m_0 c}{\hbar'}.$$

Dans le système propre, nous avons la possibilité de définir des fonctions d'ondes doublement périodiques, paires et impaires, se réduisant respectivement à $\sin \tau$ et $\cos \tau$ pour $k = 0$ suivant deux choix

- a) soit $\operatorname{sn}(\tau, k)$ et $\operatorname{cd}(\tau, k)$,
 b) soit $\operatorname{cn}(\tau, k)$ et $\operatorname{sd}(\tau, k)$.

En outre, nous pouvons définir une fonction d'ondes doublement périodique se réduisant pour $k = 0$ à une constante en considérant les fonctions

- c) $\operatorname{dn}(\tau, k)$ et $\operatorname{nd}(\tau, k)$.

Nous allons maintenant examiner les équations différentielles du second ordre que le choix de ces fonctions nous conduit à adopter pour généraliser l'équation

$$\frac{d^2}{d\tau^2} \psi(\tau) + \psi(\tau) = 0.$$

Pour cela nous examinerons les équations différentielles du second ordre dont les solutions sont les fonctions elliptiques de Jacobi.

A) L'équation

$$y'^2 + (1 - 2k^2)y^2 + k^2y^4 - k'^2 = 0,$$

a pour solutions

$$y = \operatorname{cn} u \quad \text{si} \quad y(0) = 1,$$

$$y = k' \operatorname{sd} u \quad \text{si} \quad y(0) = 0.$$

Par suite

$$y'' + (1 - 2k^2)y + 2k^2y^3 = 0,$$

a pour solutions

$$y = \operatorname{cn} u \quad \text{si} \quad y(0) = 1, \quad y'(0) = 0,$$

$$y = k' \operatorname{sd} u \quad \text{si} \quad y(0) = 0, \quad y'(0) = k'.$$

$$B) \quad y'^2 + (1 + k^2)y^2 - k^2y^4 - 1 = 0,$$

a pour solutions

$$y = \operatorname{sn} u \quad \text{si} \quad y(0) = 0,$$

$$y = \operatorname{cd} u \quad \text{si} \quad y(0) = 1.$$

Par suite

$$y'' + (1 + k^2)y - 2k^2y^3 = 0,$$

a pour solutions

$$y = \operatorname{sn} u \quad \text{pour} \quad y(0) = 0, \quad y'(0) = 1,$$

$$y = \operatorname{cd} u \quad \text{pour} \quad y(0) = 1, \quad y'(0) = 0.$$

C) L'équation

$$y'^2 - (1 + k'^2)y^2 + y^4 + k'^2 = 0,$$

a pour solutions

$$y = \operatorname{dn} u \quad \text{pour} \quad y(0) = 1,$$

$$y = k' \operatorname{nd} u \quad \text{pour} \quad y(0) = k'.$$

Par suite,

$$y'' - (1 + k^2)y + 2y^3 = 0,$$

a pour solutions

$$y = \operatorname{dn} u \quad \text{pour} \quad y(0) = 1, \quad y'(0) = 0,$$

$$y = k' \operatorname{nd} u \quad \text{pour} \quad y(0) = k', \quad y'(0) = 0.$$

Revenant des équations différentielles vérifiées par les fonctions $\psi(\tau)$ aux équations aux dérivées partielles déterminant les fonctions $\psi(x, y, z, t)$ on voit immédiatement par correspondance que

$$A) \quad (27) \quad \begin{cases} \psi_0 = \lambda \operatorname{cn} [(Kct - (\mathbf{K}\mathbf{x})), k], \\ \psi_s = \lambda k' \operatorname{sd} [(Kct - (\mathbf{K}\mathbf{x})), k], \end{cases}$$

sont solutions particulières de

$$(28) \quad \square \psi + (1 - 2k^2)\mu_0^2 \psi + \frac{2k^2\mu_0^2}{\lambda^2} \psi^3 = 0,$$

$$(\mu_0^2 = K^2 - |\mathbf{K}|^2).$$

$$B) \quad (29) \quad \begin{cases} \psi_s = \lambda \operatorname{sn} [(Kct - (\mathbf{K}\mathbf{x})), k], \\ \psi_c = \lambda \operatorname{cd} [(Kct - (\mathbf{K}\mathbf{x})), k], \end{cases}$$

sont solutions particulières de

$$(30) \quad \square \psi + (1 + k^2) \mu_0^2 \psi - \frac{2k^2 \mu_0^2}{\lambda^2} \psi^3 = 0.$$

$$C) \quad (31) \quad \begin{cases} \psi_{\operatorname{dn}} = \lambda \operatorname{dn} [(Kct - (\mathbf{K}\mathbf{x})), k], \\ \psi_{\operatorname{nd}} = \lambda k' \operatorname{nd} [(Kct - (\mathbf{K}\mathbf{x})), k], \end{cases}$$

sont solutions particulières de

$$(32) \quad \square \psi - (1 + k'^2) \mu_0^2 \psi + \frac{2\mu_0^2}{\lambda^2} \psi^3 = 0.$$

Les équations (28), (30), (32) ont été déjà rencontrées par de nombreux auteurs notamment L. SCHIFF [20], N. ROSEN et H. B. ROSENTOCK [19], R. FINKELSTEIN, R. LE LEVIER et M. RUDERMAN [6], B. J. MALENKA [13], D. IVANENKO [11].

Ces équations s'écrivent d'une façon générale

$$(33) \quad \square \psi + \alpha \psi + \gamma \psi^3 = 0,$$

α et γ désignant deux constantes.

4. - Les solutions ondes planes des équations d'ondes non linéaires $\square \psi + \alpha \psi + \gamma \psi^3 = 0$.

Réciproquement nous allons utiliser les résultats ci-dessus pour caractériser les solutions ondes planes des équations (33) que nous répartirons en quatre types

$$(34) \quad \begin{cases} (A) \quad \square \psi + \mu_1^2 \psi + \mu_2^2 \psi^3 = 0, \\ (B) \quad \square \psi + \mu_1^2 \psi - \mu_2^2 \psi^3 = 0, \\ (C) \quad \square \psi - \mu_1^2 \psi + \mu_2^2 \psi^3 = 0, \\ (D) \quad \square \psi - \mu_1^2 \psi - \mu_2^2 \psi^3 = 0. \end{cases}$$

Plus précisément, nous allons déterminer lorsqu'elles existent sous des conditions que nous préciserons les solutions de ces équations du type ondes planes d'amplitudes bornées.

A) L'équation (A) admet pour solutions les ondes planes

$$(35) \quad \psi = \lambda \operatorname{cn} [(Ket - (\mathbf{K}\mathbf{x})) + \xi_0, k],$$

avec

$$K^2 - |\mathbf{K}|^2 = \mu_0^2,$$

$$\mu_0 = \frac{m_0 c}{\hbar},$$

en déterminant μ_0 et k par

$$(1 - 2k^2)\mu_0^2 = \mu_1^2, \quad \frac{2k^2\mu_0^2}{\lambda^2} = \mu_2^2,$$

d'où

$$(36) \quad \begin{cases} \mu_0^2 = \mu_1^2 + \mu_2^2 \lambda^2, \\ k^2 = \frac{\mu_2^2 \lambda^2}{2(\mu_1^2 + \mu_2^2 \lambda^2)}. \end{cases}$$

Ici on a toujours $0 \leq k^2 \leq \frac{1}{2}$. L'onde plane n'est jamais apériodique. La masse dynamique réduite μ_0 est toujours supérieure à μ_1 tandis que la masse dynamique vraie m_0 a pour valeur

$$m_0 = \frac{\hbar \mu_0}{4cK(k)}.$$

Si $\mu_0 > \mu_1$ est fixé, λ et k sont déterminés par

$$(37) \quad \lambda^2 = \frac{\mu_0^2 - \mu_1^2}{\mu_2^2}, \quad k^2 = \frac{\mu_0^2 - \mu_1^2}{2\mu_0^2}.$$

Si au lieu de μ_0 , m_0 est donné la détermination de k est plus complexe: on devra dans ce cas résoudre l'équation transcendante

$$(38) \quad (1 - 2k^2)K^2(k) = \frac{\hbar^2 \mu_1^2}{16m_0^2 c^2}.$$

Si l'on se donne les trois constantes μ_1 , μ_2 et k , ($0 \leq k^2 \leq \frac{1}{2}$) alors

$$(39) \quad \begin{cases} \lambda^2 = \frac{2k^2 \mu_1^2}{\mu_2^2 (1 - 2k^2)}, \\ \mu_0^2 = \frac{\mu_1^2}{1 - 2k^2} \quad \text{et} \quad m_0^2 c^2 = \frac{\mu_1^2}{16K^2(1 - 2k^2)}. \end{cases}$$

Les ondes planes (35) sont solutions de (A) quelques soient les conditions initiales $\psi(0)$ ou $\psi'(0)$. Pour $\xi_0 = 0$, $\psi = \lambda \operatorname{en} \tau$, pour $\xi_0 = K$, $\psi = \lambda k' \operatorname{sd} \tau$.

B) Les équations de la forme

$$(34B) \quad \square \psi + \mu_1^2 \psi + \mu_2^2 \psi^3 = 0,$$

admettent pour solutions les ondes planes d'amplitudes bornées

$$(40) \quad \psi = \lambda \operatorname{sn} [(Kct - (\mathbf{K}\mathbf{x})) + \xi_0, k], \quad K^2 - |\mathbf{K}|^2 = \mu_0^2,$$

avec ici

$$(41) \quad \begin{cases} \mu_0^2 = \mu_1^2 - \frac{\mu_2^2 \lambda^2}{2}, \\ k^2 = \frac{\mu_2^2 \lambda^2}{2\mu_1^2 - \mu_2^2 \lambda^2}. \end{cases}$$

La condition $0 \leq k^2 < 1$ entraîne

$$0 \leq \lambda^2 \leq \frac{\mu_1^2}{\mu_2^2}.$$

Ceci correspond à des restrictions sur les données initiales.

En effet la solution ci-dessus n'existe que si les conditions initiales satisfont aux conditions

$$(\psi'(0))^2 \leq \frac{\mu_1^2}{2\mu_2^2},$$

et

$$(\psi(0))^2 \leq \frac{1}{\mu_2^2} [\mu_1^2 - \sqrt{2\mu_2^2 (\psi'(0))^2}].$$

Si ces conditions ne sont pas satisfaites, les solutions ondes planes de (B) sont des fonctions elliptiques de Jacobi devenant périodiquement non bornées et il ne semble pas que de telles fonctions soient susceptibles de représenter une structure corpusculaire physiquement réalisable.

Réciproquement la donnée de μ_0^2 , μ_1^2 , μ_2^2 détermine λ^2 et k^2 par

$$(42) \quad \begin{cases} \lambda^2 = \frac{2(\mu_0^2 - \mu_1^2)}{\mu_0^2}, \\ k^2 = \frac{\mu_1^2 - \mu_0^2}{\mu_0^2}. \end{cases}$$

Cette solution devient apériodique pour $\lambda^2 = \mu_1^2/\mu_2^2$. Alors $\mu_0^2 = \mu_1^2/2$ et

$$(43) \quad \psi_s = \frac{\mu_1}{\mu_2} \operatorname{tgh} (Kct - (\mathbf{K}\mathbf{x})), \quad \psi_c = \frac{\mu_1}{\mu_2}.$$

La relation $m_0 c = \hbar \mu_0 / 4K$ montre alors que si $\mu_0 = \mu_1 / \sqrt{2}$ reste fini, $K(1) \rightarrow \infty$, la masse propre dynamique m_0 tend vers zéro.

C) Les équations du type (C)

$$(34C) \quad \square \psi - \mu_1^2 \psi + \mu_2^2 \psi^3 = 0,$$

admettent quelque soient les conditions initiales des solutions ondes planes soit du type $\lambda \operatorname{cn} \tau$ soit du type $\lambda \operatorname{dn} \tau$.

— C_1 —

$$(44) \quad \psi = \lambda \operatorname{dn} [(Kct - (\mathbf{K}\mathbf{x})) + \xi_0, k],$$

satisfait aux équations (C), μ_0^2 et k^2 étant déterminés par

$$(45) \quad \mu_0^2 = \frac{\mu_2^2 \lambda^2}{2}, \quad k^2 = \frac{2(\mu_2^2 \lambda^2 - \mu_1^2)}{\mu_2^2 \lambda^2},$$

sous la condition

$$\frac{\mu_1^2}{\mu_2^2} \leq \lambda^2 \leq \frac{2\mu_1^2}{\mu_2^2}.$$

Pour $\xi_0 = 0$, $\psi = \lambda \operatorname{dn} \tau$; pour $\xi_0 = K$, $\psi = \lambda k' \operatorname{nd} \tau$.

Pour $|\lambda| = \mu_1/\mu_2$, $k = 0$, ψ se réduit à une constante: $\psi = \mu_1/\mu_2$.

Pour $k^2 = 1$ soit $|\lambda| = \mu_1 \sqrt{2}/\mu_2$, ψ devient apériodique

$$\left(\operatorname{dn} (u, 1) = \frac{1}{\cosh u} \right),$$

mais alors $\mu_0^2 = \mu_1^2$, $k' \rightarrow 0$. Il faut que la masse propre dynamique m_0 tende vers zéro.

Réciproquement si μ_0^2 , μ_1^2 , μ_2^2 sont données

$$\lambda^2 = \frac{2\mu_0^2}{\mu_2^2}, \quad k^2 = \frac{2\mu_0^2 - \mu_1^2}{\mu_0^2},$$

sous la condition $\mu_1^2/2 \leq \mu_0^2 \leq \mu_1^2$.

— C_2 —

$$(46) \quad \psi = \lambda \operatorname{cn} [(Kct - (\mathbf{K}\mathbf{x})) + \xi_0, k],$$

avec

$$\frac{1}{2} \leq k^2 \leq 1,$$

est solution de (C).

(Pour $\xi_0 = 0$, $\psi_c = \lambda \operatorname{cn} \tau$, pour $\xi_0 = \mp K$, $\psi_s = \pm \lambda k' \operatorname{sd} \tau$).

μ_0^2 et k^2 sont alors déterminés par

$$(47) \quad \begin{cases} \mu_0^2 = \mu_2^2 \lambda^2 - \mu_1^2, \\ k^2 = \frac{\mu_2^2 \lambda^2}{2(\mu_2^2 \lambda^2 - \mu_1^2)}, \end{cases}$$

sous la condition

$$\lambda^2 \geq \frac{2\mu_1^2}{\mu_2^2}.$$

Pour $\lambda^2 = 2\mu_1^2/\mu_2^2$, $k^2 = 1$, ψ devient apériodique

$$(48) \quad \psi_c = \frac{\mu_1 \sqrt{2}}{\mu_2} \frac{1}{\cosh \tau}.$$

Réciproquement si μ_0^2 , μ_1^2 , μ_2^2 sont donnés

$$(49) \quad \lambda^2 = \frac{\mu_0^2 + \mu_1^2}{\mu_2^2}, \quad k^2 = \frac{\mu_0^2 + \mu_1^2}{2\mu_0^2}, \quad (\mu_0^2 \geq \mu_1^2).$$

D) Les équations du type

$$(34D) \quad \square \psi - \mu_1^2 \psi - \mu_2^2 \psi^3 = 0,$$

n'admettent pas de solutions ondes planes d'amplitudes bornées.

En effet les solutions de l'équation différentielle associée

$$\frac{d^2 \psi(\tau)}{d\tau^2} - \mu_1^2 \psi - \mu_2^2 \psi^3 = 0,$$

sont suivant les conditions initiales de l'une des formes

$$\lambda \operatorname{tn} \tau, \quad \lambda \frac{\operatorname{sn} \tau}{\operatorname{cd} \tau}, \quad \lambda \operatorname{nc} \tau.$$

Ces fonctions doublement périodiques deviennent périodiquement non bornées et ne sont pas physiquement acceptables. Ceci conduit à écarter les équations du type (D).

Dans les trois cas (A), (B), (C) nous avons obtenu des solutions ondes planes du type stationnaire.

La mécanique ondulatoire linéaire considère surtout les ondes planes du type « progressif » c'est à dire de la forme

$$\psi = A \exp [i(Ket - (\mathbf{K}\mathbf{x}))],$$

correspondant aux solutions

$$\psi = A \exp [\pm i\tau], \quad (\tau = Ket - (\mathbf{K}\mathbf{x})),$$

de

$$\frac{d^2\psi}{d\tau^2} + \psi(\tau) = 0.$$

On peut se proposer de déterminer pour (34A), (34B), (34C) des ondes du même type se réduisant pour $\mu_2 = 0$ aux fonctions $A \exp [\pm i\tau]$.

Si l'on considère l'équation

$$y''_{xx} + \omega^2 y(x) = 0,$$

l'intégration directe donne

$$y'^2 + \omega^2 y^2 = \chi_0 = \text{const.}$$

$\chi_0 \neq 0$ conduit aux ondes stationnaires $\sqrt{\chi_0/\omega^2} \sin \omega x$ et $\sqrt{\chi_0/\omega^2} \cos \omega x$ tandis que $\chi_0 = 0$ conduit à $y = A \exp [\pm i\omega x]$.

Ici l'équation différentielle associée à l'équation (34A) par exemple soit

$$\psi''_{\tau^2} + \mu_1^2 \psi + \mu_2^2 \psi^3 = 0,$$

donne par intégration directe

$$(\psi'_{\tau})^2 + \mu_1^2 \psi^2 + \frac{\mu_2^2}{2} \psi^4 = \chi_0.$$

$\chi_0 \neq 0$ conduit aux solutions réelles stationnaires considérées précédemment.

$\chi_0 = 0$ conduit à un autre type de solutions.

Posant $\psi = 1/\chi$ on voit facilement que si $\chi_0 = 0$

$$(50) \quad \psi(\tau) = \frac{1}{C_1 \exp [i\mu_1 \tau] - C_2 \exp [-i\mu_1 \tau]},$$

C_1 et C_2 désignant deux constantes liées par la relation

$$(51) \quad C_1 C_2 = \frac{\mu_2^2}{8\mu_1^2}.$$

Posant $1/C_1 = \lambda_1$, $1/C_2 = -\lambda_2$, on écrit encore

$$(52) \quad \psi(\tau) = \frac{\lambda_1}{\exp[i\mu_1\tau] - (\mu_2^2/8\mu_1^2)\lambda_1^2 \exp[-i\mu_1\tau]} = \frac{\lambda_2}{\exp[-i\mu_1\tau] - (\mu_2^2/8\mu_1^2)\lambda_2^2 \exp[i\mu_1\tau]},$$

ou encore

$$(53) \quad \psi(\tau) = \frac{\lambda_1}{\exp[i\mu_1\tau](1 + \mu_2^2\lambda_1^2/8\mu_1^2) - (\mu_2^2/4\mu_1^2)\lambda_1^2 \cos \mu_1\tau} = \frac{\lambda_2}{[1 + (\mu_2^2/8\mu_1^2)\lambda_2^2] \exp[-i\mu_1\tau] - (\mu_2^2/4\mu_1^2)\lambda_2^2 \cos \mu_1\tau}.$$

Ces fonctions sont simplement périodiques. On voit facilement sur ces expressions comment s'opère lorsque $\mu_2 \rightarrow 0$ le passage aux solutions du cas de Klein-Gordon.

L'onde plane $\psi(Kct - (\mathbf{K}\mathbf{x}))$, ($K^2 - |\mathbf{K}|^2 = \mu_1^2$) n'est jamais purement progressive. À côté du terme progressif figure un terme stationnaire. Ceci peut encore s'interpréter en disant que les ondes planes de ce type ne sont jamais uniquement à énergie positive ou uniquement à énergie négative. Un terme de battement accompagne toujours le terme principal progressif à énergie positive ou négative.

5. — La composition des fonctions d'ondes dans les théories non linéaires.

Les équations (34A), (34B), (34C) ne sont pas linéaires et la somme de deux solutions n'est pas une solution. Néanmoins et c'est là un point sur lequel je veux insister maintenant, il existe pour les ondes planes solutions de ces équations un théorème d'addition ou si l'on préfère un théorème de composition.

Celui-ci va résulter immédiatement des théorèmes d'addition des fonctions elliptiques.

Considérons la fonction en u . On a vu que

$$\text{en}(u + K) = -k' \text{sd } u, \quad \text{sd}(u + K) = \frac{1}{k'} \text{en } u.$$

On peut montrer que les théorèmes d'addition des fonctions elliptiques tels qu'ils sont donnés dans les traités classiques prennent également la forme

suivante

$$(54) \quad \left\{ \begin{array}{l} \operatorname{cn}(u \pm v) = \frac{\operatorname{cn} u \operatorname{cn} v \mp k'^2 \operatorname{sd} u \operatorname{sd} v}{1 \pm k^2 \operatorname{cn} u \operatorname{sd} v \operatorname{cn} v \operatorname{sd} v}, \\ \operatorname{sd}(u \pm v) = \frac{\operatorname{sd} u \operatorname{cn} v \pm \operatorname{sd} v \operatorname{cn} u}{1 \mp k^2 \operatorname{sd} u \operatorname{cn} u \operatorname{sd} v \operatorname{cn} v}, \\ \operatorname{sn}(u \pm v) = \frac{\operatorname{sn} u \operatorname{cd} v \pm \operatorname{cd} u \operatorname{sn} v}{1 \mp k^2 \operatorname{sn} u \operatorname{cd} v \operatorname{sn} v \operatorname{cd} v}, \\ \operatorname{cd}(u \pm v) = \frac{\operatorname{cd} u \operatorname{cd} v \mp \operatorname{sn} u \operatorname{sn} v}{1 \pm k^2 \operatorname{sn} u \operatorname{cd} v \operatorname{sn} v \operatorname{cd} v}. \end{array} \right.$$

Si l'on considère pour un corpuscule représenté par l'équation (34A) des états τ_1 et τ_2 auxquels correspondent les fonctions d'ondes

$$(55) \quad \left\{ \begin{array}{l} \psi_c^{(1)} = \lambda \operatorname{cn} \tau_1, \quad \psi_s^{(1)} = \lambda k' \operatorname{sd} \tau_1, \\ \psi_c^{(2)} = \lambda \operatorname{cn} \tau_2, \quad \psi_s^{(2)} = \lambda k' \operatorname{sd} \tau_2, \end{array} \right.$$

à la fonction d'état (1) + (2) ou $\psi(\tau_1 + \tau_2)$ correspondent les fonctions

$$(56) \quad \left\{ \begin{array}{l} \psi_c^{(1)+(2)} = \lambda \operatorname{cn}(\tau_1 + \tau_2, k), \\ \psi_s^{(1)+(2)} = \lambda k' \operatorname{sd}(\tau_1 + \tau_2, k). \end{array} \right.$$

Le théorème d'addition donne alors

$$(57) \quad \left\{ \begin{array}{l} \psi_c^{(1)+(2)} = \frac{\lambda^3 [\psi_c^{(1)} \psi_c^{(2)} - \psi_s^{(1)} \psi_s^{(2)}]}{\lambda^4 + (k^2/k'^2) \psi_s^{(1)} \psi_s^{(2)} \psi_c^{(1)} \psi_c^{(2)}}, \\ \psi_s^{(1)+(2)} = \frac{\lambda^3 [\psi_s^{(1)} \psi_c^{(2)} + \psi_s^{(2)} \psi_c^{(1)}]}{\lambda^4 - (k^2/k'^2) \psi_s^{(1)} \psi_s^{(2)} \psi_c^{(1)} \psi_c^{(2)}}. \end{array} \right.$$

La possibilité de construire des fonctions d'état à deux corpuscules à partir des fonctions d'états à un corpuscule rend possible la construction d'un espace d'états nécessaire pour introduire une seconde quantification.

La seconde quantification a été généralement considérée comme nécessitant une théorie linéaire. Il me semble que ceci n'est pas nécessaire mais que la seconde quantification est essentiellement attachée à la possibilité de construire des états à 2, 3, ... n particules à partir des états à une particule. Pour cela, il suffit que dans la théorie considérée, il existe un théorème d'ad-

dition ou de composition des états, c'est à dire qu'à partir des fonctions représentant un état à n particules et un état à une particule on puisse construire un état à $n+1$ particules.

Les fonctions d'ondes acceptables seront donc celles admettant un théorème d'addition. Cette condition, nécessaire mais non suffisante, semble *a priori* très large. Néanmoins nous allons voir que l'on peut apporter à la détermination de ces fonctions une solution particulière remarquable.

En effet, WEIERSTRASS a démontré un théorème remarquable (voir par exemple le traité des fonctions elliptiques de HANCKOCK [10]) qui répond à notre question.

WEIERSTRASS appelle théorème d'addition algébrique une relation algébrique liant les fonctions $\Phi(u)$, $\Phi(v)$, $\Phi(u+v)$, et voici son théorème:

« Toute fonction pour laquelle il existe un théorème d'addition algébrique est une fonction elliptique ou l'une de ses dégénérescences ».

L'application de ce théorème aux solutions ondes planes nous conduit d'une façon limitative aux équations d'ondes considérées ci-dessus.

Toutefois la nature algébrique d'un théorème d'addition des fonctions d'ondes ne s'impose pas du point de vue de l'interprétation physique et rien ne nous conduit à penser que la nature obéisse à des règles traduites par des lois algébriques.

Je vais d'ailleurs considérer maintenant un exemple simple d'équations d'ondes généralisant les équations précédentes et pour lequel il existera un théorème d'addition non algébrique pour les ondes planes.

Pour cela je considère les équations d'ondes non linéaires

$$(58) \quad \begin{cases} (\alpha) & \square\psi + \mu_1^2 \sin \psi = 0, \\ (\beta) & \square\psi + \mu_1^2 \sinh \psi = 0. \end{cases}$$

Si l'on considère que ces équations sont « approchées » par les équations obtenues en remplaçant $\sin \psi$ et $\sinh \psi$ par les premiers termes de leurs développements en séries ces équations sont les généralisations de

$$(59) \quad \begin{cases} (\alpha') & \square\psi + \mu_1^2\psi - \frac{\mu_1^2}{6}\psi^3 = 0, \\ (\beta') & \square\psi + \mu_1^2\psi + \frac{\mu_1^2}{6}\psi^3 = 0. \end{cases}$$

Si l'on pose alors $\psi = \lambda\varphi$, $\mu_1^2\lambda^2/6 = \mu_2^2$, on obtient pour φ les équations

$$(60) \quad \begin{cases} (\alpha'') & \square\varphi + \mu_1^2\varphi - \mu_2^2\varphi^3 = 0, \\ (\beta'') & \square\varphi + \mu_1^2\varphi + \mu_2^2\varphi^3 = 0. \end{cases}$$

On retrouve les équations des types (34A) et (34B) précédents.

Les solutions du type « ondes planes » des équations (58 α) et (58 β) peuvent s'obtenir sans difficulté.

Si l'on pose

$$\tau = Kct - (\mathbf{K}\mathbf{x}), \quad \text{avec } K^2 - |\mathbf{K}|^2 = \mu_0^2,$$

les solutions « ondes planes » de (58 α) et (58 β) seront de la forme

$$\psi(\mathbf{x}, t) = \psi(\tau),$$

$\psi(\tau)$ étant solution des équations *différentielles*

$$(61) \quad \begin{cases} \frac{d^2\psi(\tau)}{d\tau^2} + \frac{\mu_1^2}{\mu_0^2} \sin \psi(\tau) = 0, \\ \frac{d^2\psi(\tau)}{d\tau^2} + \frac{\mu_1^2}{\mu_0^2} \sinh \psi(\tau) = 0, \end{cases}$$

ou

$$(62) \quad \begin{cases} (\alpha) & \psi''_{\tau^2} + \chi_1 \sin \psi(\tau) = 0, \\ (\beta) & \psi''_{\tau^2} + \chi_1 \sinh \psi(\tau) = 0. \end{cases}$$

Nous allons examiner les solutions de (62 α), celles de (62 β) s'obtenant par une analyse parallèle.

L'équation (62 α) est bien connue en physique: c'est l'équation du mouvement pendulaire:

Alors que l'équation de Klein-Gordon associait au corpuscule dans son système propre le mouvement d'un oscillateur sinusoïdal, les équations non linéaires considérées ici lui associent un mouvement pendulaire.

Les solutions de

$$(58\alpha) \quad \square\psi + \mu_1^2 \sin \psi = 0,$$

ne sont définies qu'à un multiple de 2π près. Si $\psi_0(\tau)$ est solution il en sera de même de

$$\psi_1(\tau) = \psi_0(\tau) + 2n\pi.$$

De même si l'on pose

$$\psi_2(\tau) = \psi_0(\tau) \pm \frac{n\pi}{2},$$

les fonctions $\psi_2(\tau)$ satisfont à

$$(58\gamma) \quad \square \psi_2(\tau) + \mu_1^2 \cos \psi_2(\tau) = 0.$$

Les solutions de (58 α) permettent donc d'écrire immédiatement celles de (58 β) et de (58 γ)

Pour obtenir les solutions ondes planes de (58 α) il nous suffit de considérer l'équation différentielle (62 α) qui par intégration directe donne

$$(63) \quad (\psi'_\tau)^2 - 2\chi_1 \cos \psi = \chi_0,$$

χ_0 étant une constante telle que

$$\chi_0 = \psi_0'^2 - 2\chi_1 \cos \psi_0.$$

Nous en déduisons

$$(64) \quad (\psi'_\tau)^2 = (\chi_0 + 2\chi_1) \left[1 - \frac{4\chi_1}{\chi_0 + 2\chi_1} \sin^2 \frac{\psi}{2} \right],$$

et ceci nous conduit à considérer deux cas

$$1) \quad \frac{4\chi_1}{\chi_0 + 2\chi_1} \leq 1 \quad \text{soit} \quad \chi_0 \geq 2\chi_1.$$

Posant $k^2 = 4\chi_1/(\chi_0 + 2\chi_1)$,

$$(65) \quad (\psi'_\tau)^2 = \frac{4\chi_1}{k^2} \left[1 - k^2 \sin^2 \frac{\psi}{2} \right],$$

$$2) \quad \frac{4\chi_1}{\chi_0 + 2\chi_1} = k_1^2 > 1,$$

d'où

$$(67) \quad (\psi'_\tau)^2 = \frac{4\chi_1}{k_1^2} \left[1 - k_1^2 \sin^2 \frac{\psi}{2} \right].$$

Dans le premier cas, on a immédiatement

$$2 \int_0^{\psi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \pm \frac{2\sqrt{\chi_1}}{k} \tau + 2\xi_0,$$

ou

$$F\left(\frac{\psi}{2}, k\right) = \pm \frac{\sqrt{\chi_1}}{k} \tau + \xi_0,$$

$F(\varphi, k)$ désignant l'intégrale elliptique de Legendre

$$F(\varphi, k) = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

Introduisant la fonction $\operatorname{am}(u, k) = \varphi$ telle que

$$\sin \varphi = \operatorname{sn}(u, k) \quad \text{et} \quad \cos \varphi = \operatorname{cn}(u, k),$$

on obtient

$$(68) \quad \frac{\psi}{2} = \operatorname{am}\left(\pm \frac{\sqrt{\chi_1}}{k} \tau + \xi_0, k\right),$$

$$(69) \quad \begin{cases} \sin \frac{\psi}{2} = \operatorname{sn}\left(\frac{\sqrt{\chi_1}}{k} \tau + \xi_0, k\right), \\ \cos \frac{\psi}{2} = \operatorname{cn}\left(\frac{\sqrt{\chi_1}}{k} \tau + \xi_0, k\right). \end{cases}$$

$0 \leq k^2 \leq 1$ entraîne la condition

$$\cos^2 \frac{\psi_0}{2} \leq \frac{\psi_0'^2}{4\chi_1}.$$

La solution du second cas pour lequel

$$\cos^2 \frac{\psi_0}{2} > \frac{\psi_0'^2}{4\chi_1},$$

ce qui exige

$$\psi_0'^2 < 4\chi_1,$$

se déduit de la solution du premier cas par la relation

$$F(\varphi, k_1) = kF(\varphi_1, k),$$

avec

$$\varphi_1 = \arcsin(k_1 \sin \varphi),$$

ou par la formule dite du module réciproque

$$\operatorname{sn}(ku, k_1) = k \operatorname{sn}(u, k),$$

qui nous donne ici

$$(70) \quad \sin \frac{\psi}{2} = k \operatorname{sn} (\sqrt{\chi_1} \tau + \xi_1, k).$$

Nous avons donc dans le cas considéré des expressions simples au moyen des fonctions elliptiques des ondes planes solutions des équations d'ondes (58).

Il existe encore ici un théorème d'addition pour les fonctions d'ondes solutions du type ondes planes.

En effet, soit

$$\tau_1 = K_1 ct - (\mathbf{K}_1 \mathbf{x}), \quad \tau_2 = K_2 ct - (\mathbf{K}_2 \mathbf{x}),$$

avec

$$K_1^2 - (\mathbf{K}_1)^2 = K_2^2 - (\mathbf{K}_2)^2 = \mu_0^2,$$

$\psi(\tau_1)$ et $\psi(\tau_2)$ désignant les solutions précédentes, $\psi(\tau_1 + \tau_2)$ s'exprime au moyen de $\psi(\tau_1)$ et de $\psi(\tau_2)$.

En effet nous avons

$$\psi(\tau) = 2 \operatorname{am} \left[\frac{\sqrt{\chi_1}}{k} \tau + \xi_0 \right].$$

Le théorème d'addition des fonctions $\operatorname{am} u$ nous donne

$$(71) \quad \begin{aligned} \operatorname{am} (u_1 \pm u_2) &= \operatorname{arctg} (\operatorname{tn} u_1 \operatorname{dn} u_2) \pm \operatorname{arctg} (\operatorname{tn} u_2 \operatorname{dn} u_1) = \\ &= \operatorname{arctg} \left[\frac{\sin \varphi_1}{\cos \varphi_1} \sqrt{1 - k^2 \sin^2 \varphi_2} \right] \pm \operatorname{arctg} \left[\frac{\sin \varphi_2}{\cos \varphi_2} \sqrt{1 - k^2 \sin^2 \varphi_1} \right], \\ (\sin \varphi_1 &= \operatorname{sn} u_1, \quad \cos \varphi_1 = \operatorname{cn} u_1, \quad \sin \varphi_2 = \operatorname{sn} u_2, \quad \cos \varphi_2 = \operatorname{cn} u_2). \end{aligned}$$

On en déduit immédiatement le théorème d'addition correspondant pour les fonctions $\psi(\tau_1)$, $\psi(\tau_2)$. Il n'est pas nécessaire de souligner le caractère non algébrique de ce théorème d'addition.

Il peut être intéressant de rattacher les ondes planes solutions des équations (34A), (34B), (34C) aux développements de la théorie quantique des champs.

Ceci revient à exprimer les ondes planes du cas (34A) par exemple, de la forme

$$(72) \quad \psi(\tau) = \lambda \operatorname{cn} \tau = \lambda \operatorname{cn} [\mu_0 ct, k],$$

dans le système propre, au moyen des fonctions

$$(73) \quad A \cos \tau' = A \cos \mu'_0 ct, \quad \text{ou} \quad A \sin \tau' = A \sin \mu'_0 ct.$$

La théorie des fonctions elliptiques nous fournit immédiatement deux développements de ce type.

a) Le développement en série de Fourier des fonctions elliptiques nous donne pour en u

$$(74) \quad \text{en } u = \frac{2\pi}{Kk} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1+q^{2n-1}} \cos \left[(2n+1) \frac{\pi u}{2K} \right],$$

avec

$$q = \exp \left[-\pi \frac{K'}{K} \right].$$

Nous en déduisons

$$(75) \quad \psi(\tau) = \lambda \text{ en } (\mu_0 ct, k) = \lambda \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1+q^{2n-1}} \cos (\mu'_n ct),$$

avec

$$(76) \quad \mu'_n = (2n+1) \frac{\pi}{2K} \mu_0.$$

L'onde $\psi(\tau)$ peut être considérée comme résultant d'une série particulière d'ondes planes solutions d'équations de Klein-Gordon avec une suite de masses propres réduites μ'_n multiples impairs de la masse propre réduite

$$(77) \quad \mu'_0 = \frac{\pi}{2K} \mu_0 < \mu_0.$$

b) Le développement en produit infini des fonctions elliptiques donne pour en u :

$$(78) \quad \text{en } u = 2q^{\frac{1}{2}} k'^{\frac{1}{2}} k^{-\frac{1}{2}} \cos \frac{\pi u}{K} \cdot \prod_{n=1}^{\infty} \left[\frac{1+2q^{2n} \cos (\pi u/K) + q^{4n}}{1-2q^{2n-1} \cos (\pi u/K) + q^{4n-2}} \right].$$

Ceci nous donne

$$(79) \quad \psi(\tau) = \lambda \text{ en } [\mu_0 ct, k] = 2q^{\frac{1}{2}} k'^{\frac{1}{2}} k^{-\frac{1}{2}} \cos \mu'_0 ct \prod_{n=1}^{\infty} \left[\frac{1+2q^{2n} \cos \mu'_0 ct + q^{4n}}{1-2q^{2n-1} \cos \mu'_0 ct + q^{4n-2}} \right],$$

avec ici

$$(80) \quad \mu'_0 = \frac{\pi}{K} \mu_0,$$

d'où

$$\begin{aligned} \mu'_0 &\geq \mu_0, & \text{pour } \pi/2 \leq K(k) < \pi, \\ \mu'_0 &< \mu_0. & \text{pour } K(k) > \pi: \end{aligned}$$

L'onde plane $\psi(\tau)$ s'exprime donc au moyen d'un produit infini de combinaisons d'ondes planes solutions d'une équation de Klein-Gordon pour un corpuscule de masse propre réduite $\mu'_0 = (\pi/K)\mu_0$.

6. — Solutions invariantes et solutions radiales des équations précédentes.

Nous compléterons cette étude en examinant brièvement pour les équations du type

$$(34) \quad \square\psi + \mu_1^2\psi \pm \mu_2^2\psi^3 = 0,$$

les solutions du type « ondes invariantes » $\psi(u)$ avec $u^2 = c^2t^2 - (x^2 + y^2 + z^2)$, et les solutions du type $\psi = \psi(r)$ avec $r^2 = x^2 + y^2 + z^2$.

Ces solutions particulières sont déterminées par les équations différentielles

$$(81) \quad \left[\frac{d^2}{du^2} + \frac{3}{u} \frac{d}{du} + \mu_1^2 \right] \psi(u) \pm \mu_2^2 \psi^3 = 0,$$

$$(82) \quad \left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \mu_1^2 \right] \psi(r) \mp \mu_2^2 \psi^3 = 0.$$

Les équations de ce type ont fait l'objet de nombreuses études mathématiques notamment de R. O. FORNAGUERA [7], de M. CIMINO [3] et de JAICH-NICYM [12]. Leur intégration ne semble pas rattachable à des transcendentes caractérisées jusqu'ici.

Nous n'indiquerons que quelques résultats relatifs au cas $\mu_1 = 0$.

Les équations (81) et (82) se réduisent alors à

$$(83) \quad \left[\frac{d^2}{du^2} + \frac{3}{u} \frac{d}{du} \right] \psi(u) \pm \mu_2^2 \psi^3 = 0,$$

$$(84) \quad \left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right] \psi(r) \mp \mu_2^2 \psi^3 = 0.$$

L'équation (83) admet la solution particulière remarquable

$$\psi(u) = \frac{\sqrt{\pm 1}}{\mu_2 u},$$

d'où l'on déduit l'onde invariante singulière sur le cône de lumière

$$(85) \quad \psi(x, y, z, t) = \frac{\sqrt{\pm 1}}{\mu_2 \sqrt{c^2 t^2 - (x^2 + y^2 + z^2)}}.$$

L'équation (84) sous la forme

$$\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right] \psi(r) + \mu_2^2 \psi(r) = 0,$$

se ramène à l'équation d'Emden [4] étudiée notamment par E. A. MILNE [14], N. FAIRCLOUGH [5], R. H. FOWLER [8].

En effet, si l'on pose

$$\psi(r) = \frac{1}{\mu_2} \varphi(r),$$

$\varphi(r)$ est déterminée par l'équation

$$\varphi''_{r^2} + \frac{2}{r} \varphi'_r + \varphi^3 = 0,$$

qui est la forme canonique de l'équation d'Emden adoptée par E. A. MILNE [14].

Si $\varphi(r)$ est une solution, on voit immédiatement que

$$\lambda \varphi(\lambda r),$$

est également une solution.

E. A. MILNE a étudié les différentes formes de solutions $\varphi(r)$ telles que

$$\varphi(r_0) = 0, \quad \left(\frac{d\varphi}{dr} \right)_{r=r_0} = -\frac{1}{C^{\frac{1}{2}}},$$

pour $r_0=1$ suivant les différentes valeurs de C . Il a montré notamment qu'il existe une seule intégrale positive telle que $\varphi(1)=0$ et qui pour $r=0$ prenne une valeur $\varphi(0)$ restant finie. Réciproquement (solution d'Emden) si on considère une solution $\varphi(r)$ qui pour $r=0$ prend une valeur finie (pour laquelle on peut poser $\varphi(0)=1$ avec une valeur convenable de λ) et telle que $(d\varphi/dr)_{r=r_0}=0$, on trouve la fonction tabulée par N. FAIRCLOUGH [5] qui s'annule pour $r=r_0=6.9011$ et en ce point $\varphi'(r_0)=-0.40231$ et $r_0^2\varphi'(r_0)=-2.0150$.

L'intégrale générale de

$$\varphi''_{r^2} + \frac{2}{r} \varphi'_r + \varphi^3 = 0,$$

dépend ici des deux constantes λ et C . Pour toute valeur finie de λ , il existe pour $\varphi(0)$ donné une valeur de $C=C_0$ pour laquelle il existe une solution. Cette solution s'annule pour $r=r_0$ et la tangente à cette solution pour $r=r_0$ définit $C=C_0$. Pour les autres valeurs de $C \neq C_0$ il existe des solutions $\varphi(r, C)$

telles que $\varphi(r_0, C) = \varphi(r_0, C_0)$ mais divergentes pour $r \rightarrow 0$, les unes tendant vers $+\infty$, les autres vers $-\infty$. E. A. MILNE a montré l'allure générale de ces fonctions sur un diagramme. Toutefois je ne crois pas que l'analyse de Milne ait été étendue au domaine $r > r_0$ sauf dans l'étude générale de R. O. FORNAGUERA et dans une note de JAIČNICYM dont les résultats ne semblent pas se raccorder avec ceux de Milne.

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Perturbative Expansions, Field Equations and Their Renormalization.

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CONTENTS. — 1. Introduction. — 2. Unrenormalized theory. — 3. Renormalization.

1. — Introduction.

I shall begin with a brief résumé of some results on unrenormalized field theories which are relevant for the purpose of this talk; I shall offer no proofs, since they have mostly appeared in print starting from 1952 ⁽¹⁾, and only try to establish the notation and make things as plain as possible. All this will be about *a*) perturbative expansions of propagation kernels, and *b*) equations which connect the kernels themselves. This done, one is left with a formal theory which is in good shape; in cases where there is no other trouble due to divergent integrals (approximate field theoretic models, or conceivable applications to many-body problems arising from statistical mechanics or some other parts of physics) it may even be regarded as final, because one can often shown, then, that the perturbative expansions are convergent.

In all cases of major theoretical interest, though, this is but one step to the end; just about everything turns out to be infinite, so that renormalization,

(*) Text of a talk delivered at the Collôque sur les *Problèmes Mathématiques de la Théorie Quantique des Champs*. Lille, June 3-8, 1957.

⁽¹⁾ E. R. CAIANIELLO and S. FUBINI: *Nuovo Cimento*, **9**, 1218 (1952); E. R. CAIANIELLO: *Rel. Sem. Ist. Fis. Roma* (July 1953); *Nuovo Cimento*, **10**, 1634 (1953); **11**, 492 (1954); **12**, 561 (1954); **2**, 186 (1955); **5**, 739 (1957).

or the art of removing consistently infinities into the unknown parameters of the theory, masses and charges, becomes the next major problem. The third, (but not last) question is whether in a renormalized theory solutions exist or not. Methods for carrying renormalization through flood the literature and are certainly more than I can claim to know; apart from this, regularization, or use of some device to make something finite which is not, has been much applied. Also on the question of existence much work has been done; so many people have felt the need to disprove existence that this alone shows that no one feels entirely happy with his friends' proofs: let us just say that no proof of non-existence has been proved to exist as yet.

My personal attitude is very modest: I do not know anything about existence, and offer no guesses or «intuitive» arguments, because I strongly fear that wishful thinking does not suffice to turn foul mathematics into clean physics. Also in dealing with renormalization, I am far from claiming to have solved a problem which has been solved already so many times. I rather take Hadamard's viewpoint, when he states that what matters more sometimes is not whether a problem is solved, but *how much solved* it is; and shall propose a purely mathematical version of the renormalization program (which includes, here, also regularization): one, that is, that a mathematician entirely ignorant of physics would naturally find, which nowhere makes use of infinite quantities, avoids all graphology and, above all, is simple to formulate and conducive to exact mathematical statements (majoration of integrals, etc.).

I shall show, that is, that *if the concept of integral is suitably redefined* (think, for instance, of Hadamard's *partie finie*), then it follows automatically that the formal, unrenormalized theory becomes *ipso facto* the renormalized theory. Any such prescription—infinite choices are possible—can be proved to be equivalent, to within additional *finite* renormalizations, to the formal use of counter terms. But this becomes now only a thing to prove once for all and then to forget about—we only need to know that such a proof exists, to be able to work happily with our re-defined integrals—keeping, of course, masses and charges as indeterminate parameters to be determined from experiment. Renormalization—I mean, of course, *infinite* renormalization—disappears, simply, from the stage.

I shall proceed by starting from the perturbative expansions, because the identification of the terms that the re-defined integrals eliminate with corrections to masses and charges is best seen this way; it is a trivial matter to verify, conversely, that one can work as well directly with the equations among kernels, or whatever exact or approximate equations are deduced from them—the method is, in fact, completely independent of any perturbative treatment.

Here too I shall be obliged to report only the salient points, because some hellish combinatorics is required, and you would never forgive me if I were to put all of it on the blackboard.

2. — Unrenormalized theory.

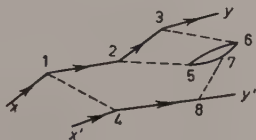
2'1. — Suppose a Fredholm equation were given to a mathematician; suppose this mathematician knew rules to evaluate all the terms which belong to the λ^n term of the Fredholm expansion of the resolvent, *but was entirely ignorant of determinants* and of how many terms are contained in the expansion of a determinant, and each term he would picture with a graph, say a term

$$\int k(x1)k(12)k(2y)k(34)k(43) \quad \text{with} \quad \begin{array}{c} \xrightarrow{x} \text{---} 1 \text{---} 2 \text{---} \xrightarrow{y} \\ \text{---} 3 \text{---} 4 \text{---} \end{array}$$

$$\int f(1) = \int d\xi_1 f(\xi_1).$$

This is exactly the situation in the theory of perturbative expansions as it was when I learned it, in 1952. Any such graph is a Feynman graph, and the case just mentioned is the very physical one of an electron in an external E.M. field; the kernel $k(xy)$ expresses the propagation of the electron from point x to point y if no field is present: we call it the « free propagator ». (The Fredholm eq. is, as everybody knows, the theory of one field) ⁽²⁾.

If two types of particles move about, one of which can create or absorb the other—say electrons and photons—then a Feynman graph will look like this:



and be described by

$$\int k(x1)k(12)k(23)k(3y)k(x'4)k(48)k(8y')k(56)k(67)k(75)h(14)h(25)h(36)h(78),$$

where $h(xy)$ is the photon free propagator, $k(xy)$ the electron free propagator.

Several papers have been published just to attempt approximate estimates of the number of such graphs pertaining to a given λ^n .

Again here we may take Hadamard's point of view: Feynman graphs solve the problem, but do not solve it « very much ». In the Fredholm theory, we could not go very far without knowing about determinants.

⁽²⁾ The best work on this subject was produced by P. T. MATTHEWS and A. SALAM: *Phys. Rev.*, **90**, 690 (1953).

It is time to state nomenclature and notation. I shall call « kernel » K what, in the Fredholm theory, is called « resolvent » or « resolving kernel », or « Fredholm minor », according to the number of variables in it. What mathematicians call kernels, I shall call « free propagators » and shall write (xy) for the electron free propagator, $[xy]$ for the photon free propagator.

$$(xy) \text{ stands for } \frac{1}{2} S_{\beta_x \alpha_y}^F(x, y) = (xy)_{\beta_x \alpha_y} \quad (\text{with spinor indices}),$$

and

$$[xy] \text{ stands for } \frac{1}{2} \delta_{\mu_x \mu_y} D^F(x, y) = [xy]_{\mu_x \mu_y} \quad (\text{with vector indices}).$$

There will be constant numerical matrices, which saturate indices of (xy) and $[xy]$: these I shall denote with $\gamma^k \equiv \gamma_{\alpha_k \beta_k}^{\mu_k}$. Thus:

$$\iint (x1) \gamma^1(12) \gamma^2(2y) [12],$$

means actually

$$\sum_{\mu, \alpha, \beta} \int d1 \int d2 (x1)_{\beta_x \alpha_1} \gamma_{\alpha_1 \beta_1}^{\mu_1}(12)_{\beta_1 \alpha_2} \gamma_{\alpha_2 \beta_2}^{\mu_2}(2y)_{\beta_2 \alpha_y} [12]_{\mu_1 \mu_2}.$$

The equations of field theory are singular, so that such expansions may very well be meaningless; but, as in the Fredholm theory, one may gain useful information by first studying the corresponding regular case, that is the formal unrenormalized theory, which we may suppose to have made regular with use of cut-offs and finite instead of infinite domains of integration.

2.2. – The Fredholm theory is only a very particular case among field theories; in general, determinants do not suffice. Given *any* field theory, I proved in 1953 ⁽¹⁾ that a perturbative expansion which generalizes that of the Fredholm resolvent can always be written if *two* algorithms are introduced: *pjaffians* and *hafnians*.

Definitions and notations. – Let $w^h = \alpha_i^h x^i$ denote a linear form (summation convention) in $x^1, x^2, \dots, x^n, \dots$ which are elements of an algebra. We define a product (outer, or Grassmann product):

$$(1) \quad x^h \wedge x^k = -x^k \wedge x^h \quad (x^h \wedge x^h = 0)$$

Then, as is well known:

$$(2) \quad w^1 \wedge w^2 \wedge \dots \wedge w^k = \frac{1}{k!} \begin{pmatrix} 1 & 2 & \dots & k \\ i_1 & i_2 & \dots & i_k \end{pmatrix} x^{i_1} \wedge x^{i_2} \wedge \dots \wedge x^{i_k}, \quad (k \leq n; = 0, \text{ if } k > n).$$

where $\begin{pmatrix} 1 & 2 & \dots & k \\ i_1 & i_2 & \dots & i_k \end{pmatrix}$ is the Sylvester notation for a determinant with row indices $1, 2, \dots, k$, column indices i_1, i_2, \dots, i_k and elements $\alpha_{i_k}^h$.

If we define another type of product (Clifford product):
(forgive my notation!)

$$(3) \quad x^h \wedge x^k = -x^k \wedge x^h + 2\delta_{h,k} \quad (x^h \wedge x^h = 1)$$

we find

$$(4) \quad w^1 \wedge w^2 \wedge \dots \wedge w^k = \sum_{r=0}^{[k/2]} \sum_{c_r} (-1)^P (h_1 \dots h_{2r}) w^{i_1} \wedge w^{i_2} \wedge \dots \wedge w^{i_{k-2r}},$$

where: $h_1 < h_2 < \dots < h_{2r}$; $i_1 < i_2 < \dots < i_{k-2r}$ is a permutation of $1, 2, \dots, k$, of parity p ; \sum is the sum over all such permutations for a given r ; $[k/2]$ is the maximum integer contained in $k/2$; $(h_1 \dots h_{2r})$ is the *pfaffian*, as defined below
If $k = 2$:

$$w^1 \wedge w^2 = (12) + w^1 \wedge w^2, \quad (12) = \alpha_i^1 \alpha_i^2.$$

Once the elements (hk) are defined (mark that, in general, $(hk) \neq (kh)$, as against the example just reported), then the *pfaffian* is defined by recursion:

$$(5) \quad (1 \dots 2r) = \sum_{h=2}^r (-1)^h (1h)(2 \dots h-1, h+1, \dots 2r), \quad ((2r-1)!! \text{ terms})$$

or

$$= \sum' (-1)^P (i_1 i_2) \dots (i_{2r-1}, i_{2r}),$$

where \sum' denotes sum over all permutations $i_1 i_2 \dots i_{2r}$ of $1, 2, \dots, 2r$ (of parity p) such that:

$$i_1 < i_2; \quad i_3 < i_4; \quad \dots; \quad i_{2r-1} < i_{2r} \quad \text{and} \quad i_1 < i_3 < \dots < i_{2r-1}.$$

A convenient symbol is also

$$(1 \dots 2r) = \begin{vmatrix} (12) & (13) & \dots & (1, 2r) \\ & (23) & \dots & (2, 2r) \\ & & \ddots & \\ & & & (2r-1, 2r) \end{vmatrix}$$

which reminds us of the well known property:

$$(1 \ 2 \ \dots \ 2m)^2 = \begin{pmatrix} 1 & 2 & \dots & 2m \\ 1 & 2 & \dots & 2m \end{pmatrix}$$

where $\begin{pmatrix} 1 & 2 & \dots & 2m \\ 1 & 2 & \dots & 2m \end{pmatrix}$ is the determinant of elements $a_{hk} = \begin{cases} (hk), & h < k, \\ -(kh), & h > k, \\ 0, & h = k. \end{cases}$

In a pfaffian, there are *lines* (line h : all elements with a label h), but nor *rows* and *columns*.

This is all I shall say about pfaffians and determinants for the time being. The actual story is much more involved with combinatorics, but I think this may suffice to show why *anticommuting fields* give rise, in the kernel expansions only to pfaffians and determinants (the latter are a particular case of the former: if suitable elements of a pfaffian are made to vanish, it reduces to a determinant of half its order).

Typical of the above is the expansion rule of pfaffians and determinants which prescribes alternating signs—a consequence of the anticommutation relations, or Fermi statistics, of fermion fields.

When studying the other accepted case, of *commuting* or boson fields, one finds that exactly the same algorithms appear, except that all terms in the expansions have always a *positive sign*. To avoid confusion, I use systematically *round brackets* for things connected with anticommutation and *square brackets* for things connected with commutation. In the study of boson fields, one finds thus:

permanents $\begin{bmatrix} 1 & \dots & m \\ 1 & \dots & m \end{bmatrix}$ of elements $[hk]$ (counterpart of determinants)

and

hafnians $[1 \dots 2m]$ of elements $[hk]$ (counterpart of pfaffians).

(The last name is my own contribution, for which I offer no apology to you, but rather a nostalgic thought to Copenhagen).

2'3. *Kernel expansions*. — I shall mention only electrodynamics (with a non-vanishing photon mass), but everything I say applies, *mutatis mutandis*, to all cases. The kernels I speak of are the vacuum expectation values of time ordered products of field operators you have heard so much about; if they are known, all is known. (They bear a variety of names in the literature, but I am quite contented with calling them kernels.)

If $\hbar = c = 1$, and λ is the interaction strength, the fundamental expansion formula in electrodynamics turns out to be:

$$(6) \quad K_{N_0 P_0} = K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0} \right) = \\ = \sum_{N(P_0)} \frac{\lambda^N}{N!} \int \sum \gamma^1 \dots \gamma^N \left(\begin{matrix} x_1 \dots x_{N_0} & \xi_1 \dots \xi_N \\ y_1 \dots y_{N_0} & \xi_1 \dots \xi_N \end{matrix} \right) [t_1 \dots t_{P_0} \xi_1 \dots \xi_N].$$

In it: N_0 in-going and N_0 out-going electron lines, P_0 photon lines — $N(P_0)$ means that $N \geq 0$ has the same parity as P_0 . The determinant is the decay product of a primeval pfaffian (in a Majorana theory, the pfaffian would survive), and contains all that pertains to electrons; the hafnian carries all the photon contribution.

If the E.M.field is not quantized, then the hafnian is not there and \sum_N ranges over *all* $N \geq 0$: we recognize the Fredholm expansion. If the particle field is not quantized, then only the hafnians are left in, and we can play around and sum the series exactly, obtaining known results of Feynman and others.

We shall see that it suffices to consider the purely fermionic kernels ($P_0 = 0$). Then the series can be written (if it is allowed to exchange integrations)

$$(7) \quad K_{N_0,0} = K \begin{pmatrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{pmatrix} = \sum_{n=0}^{\infty} \left(\frac{\lambda^2}{2} \right)^n \frac{1}{n!} \int \sum \gamma^1, \dots, \gamma^{2n} \begin{pmatrix} x_1 \dots x_{N_0} & \xi_1 \dots \xi_{2n} \\ y_1 \dots y_{N_0} & \xi_1 \dots \xi_{2n} \end{pmatrix} [\xi_1 \xi_2] \dots [\xi_{2n-1} \xi_{2n}] .$$

The vacuum expectation values are actually the ratios $K_{N_0 P_0}/K_{0,0}$, just as is the case with the Fredholm resolvent. I have also formulae ⁽³⁾ that give, as simply, the expansions of these ratios, or of such things as the $\lg K_{0,0}$:

$$\lg K_{0,0} = \sum_{n=1}^{\infty} \frac{\lambda^{2n}}{2n} \int \sum \gamma^1, \dots, \gamma^{2n} [\xi_1 \xi_2] \dots [\xi_{2n-1} \xi_{2n}] \begin{pmatrix} \xi_1 \xi_2 & \xi_3 \xi_4 & \dots & \xi_{2n-1} \xi_{2n} \\ \xi_1 \xi_2 & \xi_3 \xi_4 & \dots & \xi_{2n-1} \xi_{2n} \end{pmatrix} ,$$

where the bars in the determinant symbol indicate a simple rule to suppress some terms of the expansion. This last formula, if compared with the familiar one for the \lg of the Fredholm determinant, exhibits clearly all the similarities and all the differences between a two-field and a one-field theory.

Convergence. — It was generally believed that an expansion such as (6) or (7) (at the time in which it was a major job to write or count each term of the coefficient of λ^N) should be actually divergent: there are indeed too many terms in this coefficient, and the relative signs were not known.

By looking at (7), however, it is easy to see ⁽⁴⁾ that, if we take a non-singular theory (4-integr. vol. Ω finite; $|(xy)_{\alpha\beta\gamma}| < M$, $|[xy]_{\mu\alpha\mu\gamma}| < L$):

$$(8) \quad |K_{N_0,0}| < \sum_{n=0}^{\infty} \left(\frac{\lambda^2}{2} \right)^n \frac{4^{6n}}{n!} \Omega^{2n} L^n [M^{2n}(N_0 + 2n)^{(N_0/2)+n}] ,$$

⁽³⁾ E. R. CAIANIELLO: *Nuovo Cimento*, **2**, 186 (1955).

⁽⁴⁾ E. R. CAIANIELLO: *Nuovo Cimento*, **3**, 223 (1956).

which is immediately seen to have a *finite* radius of convergence, independent of N_0 . This result is a disproof of previous too pessimistic views rather than a proof—but may have full value in cases other than relativistic field theories.

If we specialize our restriction further, and assume that the boundedness of (xy) is a consequence of the fermion field having only a *finite* number of free modes, then it can be proved that the radius of convergence of (6) or (7) is *infinite* ⁽⁵⁾: $K_{y_0 p_0}$ is an *entire function*, and everything works as in the Fredholm theory. The proof would take too much of your time, and I have to ask you to believe YENNIE and me.

I have not pushed this study further, for lack of time: there is a variety of problems that could be studied and I should not be surprised if some interesting results could be gathered with little effort, because the actual theory is after all a limiting case of such «regular», unphysical approximations. It will also be interesting to study the effects of renormalization—with some method suggested by the results I shall discuss later—on the convergence of these expansions.

2.4. Expansions among kernels. — The kernels satisfy equations of various sorts, which I call *branching equations* because of the graphical pictures that can be associated to them. Since we have the perturbative expansions at hand, it is most simple to derive the equations from these; but it can be also immediately proved, and this has been done by many people (e.g. CANDLIN ⁽⁶⁾), including myself ⁽¹⁾, that all such equations hold true even if perturbative expansions are meaningless. These equations can be taken as the definition of the theory; they depend only on the commutation properties of the fields, and are therefore valid also for non-relativistic theories; if one starts from them, no mention need be made of the vacuum state, or of the fact that free propagators are solutions of differential equations. Such a theory may well turn out, for instance, to be different from standard field theory. It is important for the mathematician, I believe, if not for the physicist, that they are *integral equations*. The corresponding differential equations could be immediately written, and are, of course, of hyperbolic type.

I shall list some of them:

$$(9) \quad K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{p_0} \right) = \sum_{h=2}^{p_0} [t_1 t_h] K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_2 \dots t_{h-1}, t_{h+1} \dots t_{p_0} \right) + \\ + \lambda \int d\xi \gamma^\xi [t_1 \xi] K \left(\begin{matrix} x_1 \dots x_{N_0} \ \xi \\ y_1 \dots y_{N_0} \ \xi \end{matrix} \middle| t_2 \dots t_{p_0} \right),$$

⁽⁵⁾ This very interesting result was kindly communicated to me by D. R. YENNIE; an alternative proof has been given by A. BUCCAFURRI and myself, in Naples. Both papers are in print.

⁽⁶⁾ D. J. CANDLIN: *Nuovo Cimento*, **12**, 380 (1954).

which proves that it suffices to know only $K_{N_0,0}$.

$$(10) \quad K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0} \right) = \sum_{h=1}^{N_0} (-1)^{h-1} (x_1 y_h) K \left(\begin{matrix} x_2 \dots & \dots x_{N_0} \\ y_1 \dots y_{h-1}, y_{h+1} \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0} \right) +$$

$$- \lambda \int d\xi \gamma^\xi(x_1 \xi) K \left(\begin{matrix} \xi x_2 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \xi t_1 \dots t_{P_0} \right),$$

(↑ difference from Fred. eq.),

(and the associate equation)

$$(11) \quad K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \right) = \sum_{h=1}^{N_0} (-1)^{h-1} (x_1 y_h) K \left(\begin{matrix} x_2 \dots x_{N_0} \\ y \neq y_h \end{matrix} \right) - \lambda^2 \int d\xi_1 \int d\xi_2 \sum \gamma^{\xi_1} \gamma^{\xi_2} \cdot$$

$$\cdot (x_1 \xi_1) [\xi_1 \xi_2] K \left(\begin{matrix} \xi_1 \dots x_{N_0} \quad \xi_2 \\ y_1 \dots y_{N_0} \quad \xi_2 \end{matrix} \right),$$

$$(12) \quad \frac{\partial}{\partial \lambda} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0} \right) = \int d\xi \sum \gamma^\xi K \left(\begin{matrix} x_1 \dots x_{N_0} \quad \xi \\ y_1 \dots y_{N_0} \quad \xi \end{matrix} \middle| \xi t_1 \dots t_{P_0} \right),$$

$$(13) \quad \frac{\partial}{\partial m_f} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0} \right) = -i \int d\xi K \left(\begin{matrix} \xi \quad x_1 \dots x_{N_0} \\ \xi \quad y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0} \right),$$

$$(14) \quad \frac{\partial}{\partial (m_b^2)} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0} \right) = -\frac{i}{2} \int d\xi K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| \xi \xi t_1 \dots t_{P_0} \right),$$

(13) and (14) ($(\xi\xi) = [\xi\xi] = 0$, by def.) derive from:

$$(13') \quad (\gamma \partial_x + m_f)(xy) = i \delta(x-y) \rightarrow \frac{\partial(xy)}{\partial m_f} = i \int d\xi (x\xi)(\xi y),$$

and

$$(14') \quad (\square_x - m_b^2)[xy] = i \delta(x-y) \rightarrow \frac{\partial[xy]}{\partial (m_b^2)} = -i \int d\xi [x\xi][\xi y],$$

and are the counterparts of Ward's identities.

These equations can be exploited to get other equations which contain only *one* kernel (but then, an infinite number of conditions is required to get a unique solution): some of these contain $\partial/\partial\lambda$ and $\partial^2/\partial\lambda^2$ only, besides integral operators. Very interesting consequences can be deduced from them, but since work on these lines has just begun, I prefer to make no statements on this subject. It is interesting to observe that such equations, considered as differential equations in λ are singular for $\lambda = 0$ and $\lambda = \infty$. This is, indeed, one of their major attractions.

It may well be that the actual renormalized theory does not admit perturbative expansion; it can be easily seen, though (⁷), that it follows from

(⁷) E. R. CAIANIELLO: *Nuovo Cimento*, **1**, 337 (1955).

the branching equations that *if* the renormalized equations admit solutions, and *if* these solutions are continuous in the point $\lambda = 0$, then also *all* λ -derivatives of the kernels exist and are continuous for $\lambda = 0$. This is, I think, a non-trivial result.

3. - Renormalization.

3.1. - The first part of this program, as outlined above, was successful mainly because the problems were studied in their most general form, and because use was made of *configuration* rather than momentum space. This I have found to be even truer in my study of renormalization: as long as I dealt with momentum-space integrals, I could only stare at expressions which were infinite, and that was all. In configuration space, instead, it is quite easy to analyze the reasons why the integrals diverge. Two are the obvious causes of infinities: the fact that the domain of integration is infinite, and the fact that the singularities of the integrand, when two or more points coincide (or are on the light cone of each other) become of too high orders; the latter is the essential point to investigate, which remains completely hidden in momentum space.

First of all, let us get rid of the first trouble. It can appear either as an infrared divergence: we rule this out by giving the photon a finite mass, and that's that; or it can show up in terms connected with vacuum fluctuations, which it is convenient in this formalism to leave in to the very end, dividing then everything by $K_{1,0}$: according to tastes, we shall either keep until the last but one minute a finite 4-volume of integration Ω , or introduce convergence factors to be kept until division by $K_{0,0}$ is performed.

The second trouble is the root of all evil^(*). I shall start, again, from the perturbative expansions; it will be quite simple to see then that the method I propose, if it works at all, works in general.

Since singularities in the integrand occur both whether points coincide, or are on the light cone of one another, I shall use the word *confluence* to denote both situations: we shall have confluences of order 1 if a pair of variables are involved, of order h if $h+1$ variables are confluent. Clearly, the singularity is of a different type for different orders of confluence.

An essential fact is that each integrand in the perturbative expansions is a *symmetric function* of its arguments ξ_1, \dots, ξ_N : this makes legitimate the

(*) The first to understand and describe very clearly this fact was, to my knowledge, J. VALATIN. I refer to his talk for further details on his views, and just take this opportunity to thank him for several interesting discussions on this subject.

free exchange of orders of integrations, which would be highly questionable for single Feynman graphs.

The main problem is one of combinatorics. It is so complicated, that several tricks are necessary to cope with it.

3.2. An example. — Let us first consider a very simple example, which shows some of the main combinatorial aspects of the question without the additional troubles due to the occurrence of infinities.

Replace, that is, integrations with summations in the perturbative expansion of K_{N_0, P_0} , which becomes (omitting, for brevity, the x, y, t variables), (let us forget here about γ -matrices):

$$(15) \quad K_{N_0, P_0}(\lambda) = \sum_N \frac{\lambda^N}{N!} \sum_{i_1 \dots i_N} f_{i_1 \dots i_N}^{N_0, P_0},$$

where $f_{i_1 \dots i_N}^{N_0, P_0}$ is symmetric in all indices, which range all over a same set of values. We may take \sum over all N , by defining $f_{i_1 \dots i_N}^{N_0, P_0}$ to be zero if $N \neq N(P_0)$.

It is very easy to see, then, that:

$$(16) \quad \sum_{i_1 \dots i_N} f_{i_1 \dots i_N} = \sum_{h=1}^N \binom{N-1}{h-1} \sum_i \sum_{i_1 \dots i_{N-h} \neq i} f_{i \dots i \underbrace{i_1 \dots i_{N-h}}_h},$$

this being simply another way of counting terms.

Confluence now is mere coincidence of indices: confluences among indices i' are still left in, but the condition $i \neq i_1' \dots i_{N-h}'$ isolates confluences of various orders in the index i ,

$$(16)' \quad \left\{ \begin{array}{ll} \sum_i \sum_{i_1 \dots i_{N-h} \neq i} f_{i \dots i i_1 \dots i_{N-h}}, & \text{can equally well be written} \\ \sum_{i_1 \dots i_{N-h}} \sum_{i \neq i_1' \dots i_{N-h}'} f_{i \dots i i_1' \dots i_{N-h}'}, & \text{which means that for any given set} \end{array} \right.$$

of $i_1' \dots i_{N-h}'$ the index i can only be given values not taken already by some i' . (Note that $\sum_{i \neq i_1' \dots i_{N-h}'} = \sum_i'$ has a different meaning for different values of $N-h$).

Going back to the perturbative expansions, one finds, after some juggling:

$$(17) \quad \frac{\partial K(x_1 \dots x_{N_0} | y_1 \dots y_{N_0} | t_1 \dots t_{P_0})}{\partial \lambda} = \sum_{h=0}^{\infty} \frac{\lambda^h}{h!} \sum_i K' \left(x_1 \dots x_{N_0} \underbrace{i \dots i}_{h+1} \middle| y_1 \dots y_{N_0} \underbrace{i \dots i}_{h+1} t_1 \dots t_{P_0} \right),$$

where K' means a kernel evaluated with the rule: for each i , sum over all

values of indices $\neq i$. To compare with:

$$(12^*) \quad \frac{\partial K_{N_0 P_0}}{\partial \lambda} = \sum_i K \left(\begin{matrix} x_1 \dots x_{N_0} & i \\ y_1 \dots y_{N_0} & i \end{matrix} \middle| i \ t_1 \dots t_{P_0} \right).$$

We discover the remarkable and unpredictable fact, that adoption of this rule ($K \rightarrow K'$) changes (12)* into (17), so that all terms isolated away by this rule are classed in increasing order of confluence and group together again into *kernels*, whose formal properties are entirely known to us. This finding is the only reason why this line of approach proves successful.

Note that the kernels K' still have confluences among all indices $i' \neq i$. Divergencies, that is, are far from being removed at this stage. But let us suppose that all terms with $h > 1$ in (17) are dropped: either because we just impose so, or because, as will be the case, after due changes, in electrodynamics, we are able to show that their omission is equivalent to a «renormalization through counter-terms» of charges and masses: to simulate the latter case, we write, in doing so, $\bar{\lambda}$ instead of λ .

(12)* is therefore substituted by

$$(18) \quad \frac{\partial K_{N_0 P_0}}{\partial \bar{\lambda}} = \sum_i K' \left(\begin{matrix} x_1 \dots x_{N_0} & i \\ y_1 \dots y_{N_0} & i \end{matrix} \middle| i \ t_1 \dots t_{P_0} \right).$$

But (12)* generates, in an obvious way, the perturbative expansion of $K_{N_0 P_0}$. If we try to generate it from (18), we have to consider higher derivatives obtained by iteration from (18). Thus

$$\frac{\partial^2 K_{N_0 P_0}}{\partial \bar{\lambda}^2} = \sum_i \frac{\partial}{\partial \bar{\lambda}} K' \left(\begin{matrix} \dots & i \\ \dots & i \end{matrix} \middle| i \dots \right) = \sum_{i \neq j} K'' \left(\begin{matrix} x_1 \dots x_{N_0} & ij \\ y_1 \dots y_{N_0} & ij \end{matrix} \middle| ij \ t_1 \dots t_{P_0} \right),$$

where K'' means, with reference to a perturbative expansion, «summed over all indices $\neq i, j$ »: because i was already excluded from the summation range for K' , and now we have to eliminate all confluences with j ($\neq i$).

One sees that iteration to all orders automatically gives sums of this type:

$\sum_{i_1 \neq i_2 \neq i_3 \neq \dots \neq i_N}$, that is: *all terms with confluences are dropped*. If such terms are infinite, we have now a finite theory.

The use of the equation for $\partial K_{N_0 P_0} / \partial \bar{\lambda}$ is therefore only a convenient trick to avoid the simultaneous consideration of all possible *disjoint* sets of confluences, which would be an awful task (9).

(9) To my knowledge, B. FERRETTI was the first to use equations of this type in connection with renormalization: my indebtedness to him far exceeds, though, any particular question of field theory, for I have received from him more instruction and encouragement than I can possibly thank him for.

Finally, we can use (16)' to change

$$\sum_i K' \left(\begin{array}{c} \dots i \\ \dots i \end{array} \middle| i \dots \right) \quad \text{into} \quad \sum_i K \left(\begin{array}{c} \dots i \\ \dots i \end{array} \middle| i \dots \right),$$

which is now a prescription on how the summation must be performed: it tells that \sum_i must be performed first, and over all indices *not already present* in any total or *partial* expansion of K . This is the analogue of taking a *partie finie* of an integral, and is the form that we shall use later. It makes the procedure independent of perturbative expansions.

(16) is very significant: it shows that we can either, with our rule, perform an integration at a time and drop terms, or drop terms from the N -fold sum as a whole, obtaining the same result. In the last version, we find here the analogue of Salam's procedure to subtract overlapping divergencies. The latter, however, due to the use of momentum space and other reasons (it is less restrictive than ours), cannot be replaced by N successive integrations with a subtraction rule. It is most essential that both things be equivalent, or else we could make no strict mathematics at all on the branching equations:; these, after any number of iterations, still must yield, for consistency, possible equations with the same solutions (if any), and this could not happen if the rule for renormalizing an N -fold integral were not applicable also by successive steps; since it is so, this will suffice to show that the procedure is independent of perturbative expansions.

3.3. Renormalization. — There are infinite ways of obtaining finite results out of divergent integrals: the task is to show that, by so doing, one is actually performing renormalization of charges and masses. From the previous example we have learnt some combinatorial tricks, and the technique to use λ -derivative equations for kernels. We have also learnt the essential rôle of the symmetry of integrand as a whole in its arguments, and the necessity that whatever rule we use, it must satisfy (16), in some form or another, to assure iterability of the procedure.

There are infinite procedures which satisfy these requirements (other conditions shall be found later, that are also easily satisfied). They can take the most various forms: Hadamard's *partie finie*, Riesz's analytical continuation, Leray's convergence factors, and as many more as you care to invent. I shall first give a couple of examples, then proceed with a general formalism which need not specify the actual prescription which is chosen.

The simplest to exhibit—although may be not the best—is the use of convergence factors. Start again from the perturbative expansions. Each integrand is of type

$$\int \dots \int \sum \gamma^1 \dots \gamma^N \left(\begin{array}{ccc} x_1 \dots x_{N_0} & 1 \dots N \\ y_1 \dots y_{N_0} & 1 \dots N \end{array} \right) [1 \dots N \quad t_1 \dots t_{P_0}].$$

It is desirable (although not necessary) to use covariant methods. Introduce then for each x, y, t, ξ a *real number* $\varepsilon_x, \dots, \varepsilon_\xi$ and multiply the integrand by a form factor which makes it convergent, e.g. of type:

$$\exp \left[- \sum_{i,j} \frac{(\varepsilon_i - \varepsilon_j)^2}{|(u_i - u_j)^2|} \right],$$

where $u \equiv x \dots y \dots t \dots \xi_1 \dots \xi_N$.

This is symmetric, so that the integral, due to the symmetry of the integrand, is a symmetric function of ξ_1, \dots, ξ_N . (Divergencies arising because Ω is ∞ may be eliminated by multiplying by $\exp \left[\sum_{i < j} (\eta_i - \eta_j)^2 |(u_i - u_j)^2| \right]$, but this point we neglect here). The integral therefore is a function of $x \dots y \dots t \dots \varepsilon_x \dots \varepsilon_y \dots \varepsilon_t \dots \varepsilon_\xi = \varepsilon_1 \dots \varepsilon_N$, convergent except when some ε 's coincide. The $\varepsilon_x, \varepsilon_y, \varepsilon_t$ we leave arbitrary: by so doing, we shall obtain the kernel, which is a distribution, as the limit of a regular function which tends to the distribution when $\varepsilon_x \varepsilon_y \varepsilon_t \rightarrow 0$: this is an appropriate way to introduce distributions. We need only consider $\varepsilon_1 \dots \varepsilon_N$. Let the integral (omitting explicit mention of x, \dots, ε_t) be the symmetric function $= f(\varepsilon_1 \varepsilon_2 \dots \varepsilon_N)$.

We are now exactly in the situation in which formula (16) holds: confluence in 1 means some $\varepsilon = \varepsilon_1$. Suppose first that f can be expanded in a Laurent series in the variables $(\varepsilon_1 - \varepsilon_{i>1})$: for each such variable, \sum can be split into two parts, one being the *singular part*, the other the *regular part* (of which, of course, only the first term survives). The symmetry assures that no matter what i we start from, the expansion is always the same. Then, *if the additional requirements we shall meet later are fulfilled*, renormalization shall mean here: keep only the regular parts, for each $(\varepsilon_1 - \varepsilon_{i>1})$ separately.

We can as well take from the beginning $\varepsilon_1 = 0$. Once the negative powers of $(\varepsilon_i - 0)$ are eliminated, we are left with a symmetric function of $\varepsilon_2 \dots \varepsilon_N$, and we can start all over again. This is of course a general outline: in practice, one will find ways to do it better and quicker. If a Laurent expansion is not possible (we do not know), the thing works as well: in any case, it will be possible to decide, without ambiguity, how to split into regular and singular parts in $(\varepsilon_i - \varepsilon_1)$.

The proof of this statement reduces to showing that something like formula (16) holds. Consider only confluences in ε_1 , and assume Laurent-expandibility ($a_{h_2 \dots h_N}$ *symmetric*):

$$\begin{aligned} f(\varepsilon_1 \varepsilon_2 \dots \varepsilon_N) &= \left[\sum_{h_2 = -\infty}^{-1} + \sum_{h_2 = 0}^{+\infty} \right] \dots \left[\sum_{h_N = -\infty}^{-1} + \sum_{h_N = 0}^{+\infty} \right] [(\varepsilon_2 - \varepsilon_1)]^{h_2} \dots [(\varepsilon_N - \varepsilon_1)]^{h_N} = \\ &= \left[\sum_{h_2, \dots, h_N \geq 0} + (N-1) \sum_{h_2, \dots, h_N \geq 0} \sum_{h_2 = -\infty}^{-1} + \binom{N-1}{2} \sum_{h_2, \dots, h_N \geq 0} \sum_{h_2 h_3 = -\infty}^{-1} + \dots \right], \end{aligned}$$

due to the symmetry, which is (16). The overlapping terms split away quite naturally. $\{(N-1) \sum_{h_3 \dots h_N \geq 0} \sum_{h_2 = -\infty}^{-1}$ stand of course for:

$$\sum_{h_3 \dots h_N \geq 0} \sum_{h_2 < 0} a_{h_1 \dots h_N} [\{(\varepsilon_2 - \varepsilon_1)\}^{h_2} \dots \{(\varepsilon_N - \varepsilon_1)\}^{h_N} + \{(\varepsilon_2 - \varepsilon_1)\}^{h_2} \{(\varepsilon_3 - \varepsilon_1)\}^{h_3} \{(\varepsilon_N - \varepsilon_1)\}^{h_N} + \dots]$$

etc., which is the same: after *all* splittings are performed, all ε 's will tend to zero, and the numerical coefficients come out right.} The only term that survives after elimination of ε_1 confluences is $\sum_{h_3, \dots, h_N} \geq 0$.

A better procedure may be to consider the integrand as a function of $(x_i - x_j)^2$ (the $\gamma\hat{c}$ terms coming out of the Feynman fermion free propagators can be disposed of quite simply—we shall not go into this). Then, replace each $(x_i - x_j)^2$ by $(x_i - x_j)^2 - i(\varepsilon_i - \varepsilon_j)^2$, and procede as above: this is one of Hadamard's prescriptions for taking the *partie finie*. This has the advantage of changing the free propagators, which are distributions, into honest and simple analytic functions: the same process that splits off the infinities secures too that the proper limits which change the analytic functions into the distributions are taken ⁽¹⁰⁾.

In conclusion, it is better to study the question in general, without specifying the rule, except that it must produce something like (16).

3.4. Renormalization continued. — We can now extend the results obtained for our example to the general case. Write, whatever the explicit rule we decide to adopt:

$$(19) \quad \int d\xi_1 \int d\xi_2 \dots \int d\xi_N F(\xi_1 \dots \xi_N) = \int d\xi_1 \left[\int_1 d\xi_2 + d_1^2 \right] \dots \left[\int_1 d\xi_N + d_1^N \right] F(\xi_1 \dots \xi_N),$$

where

$$(20) \quad \int d\xi_k = \int_1 d\xi_k + d_1^k,$$

denotes that in performing the ξ integration *all, and only*, the divergent parts if any, which arise because ξ_k is confluent with ξ_1 are eliminated: a « *partie finie* » rule, formally, and the corresponding divergence is called $d_1^k F(\xi_1 \dots \xi_N)$. The prescription will be such (as is actually the case with the previous examples, due to the symmetry of the integrand) that $\int_1 d\xi_k d_1^k = d_1^k \int_1 d\xi_k$, etc. This is a requirement that need be satisfied only after all limits are taken (that is for infinite d_1^k , etc.).

⁽¹⁰⁾ This has been suggested by A. S. WIGHTMAN, to whom also I am greatly indebted for many enlightening discussions.

We find, therefore, the counterpart of (17):

$$(21) \quad \frac{\partial K_{N_0, P_0}}{\partial \lambda} = \sum_{h=0}^{\infty} \frac{\lambda^h}{h!} \int d\xi \sum d_{\xi}^1 \dots d_{\xi}^h \gamma^{\xi} \gamma^1 \dots \gamma^h K' \left(\begin{matrix} x_1 \dots x_{N_0} & \xi \xi_1 \dots \xi_h \\ y_1 \dots y_{N_0} & \xi \xi_1 \dots \xi_h \end{matrix} \middle| \xi \xi_1 \dots \xi_h t_1 \dots t_h \right)$$

which still contains divergencies, but not those divergencies arising from confluences with ξ . (21) is to be compared with

$$(12) \quad \frac{\partial K_{N_0, P_0}}{\partial \lambda} = \int d\xi \sum \gamma^{\xi} K \left(\begin{matrix} \dots & \xi \\ \dots & \xi \end{matrix} \middle| \xi \dots \right).$$

The next step requires some complicated combinatorics. In (21), the K' contain, if expanded, infinite elements: $(\xi_i \xi_h)$ etc. We must take all such things away, so to have them acted upon directly by the d_{ξ}^h operators. Some combinatorial theorems must be invented, after which one finds:

$$(22) \quad \frac{\partial K_{N_0, P_0}}{\partial \lambda} = \sum_{h=0}^{\infty} \frac{\lambda^h}{h!} \int d\xi \sum d_{\xi}^1 \dots d_{\xi}^h \gamma^{\xi} \gamma^1 \dots \gamma^h \sum_{r=0}^{h+1} \sum_{c_r} (-1)^p \sum_{q=0}^{[h+1/2]} \sum_{c_q} \left(\begin{matrix} \xi_{h'_1} \dots \xi_{h'_r} \\ \xi_{h'_1} \dots \xi_{h'_r} \end{matrix} \right) \cdot [\xi_{l'_1} \dots \xi_{l'_{2q}}] \cdot K' \left(\begin{matrix} \xi_{h_1^{0''}} \dots \xi_{h_s^{0''}} & x_1 \dots x_{N_0} \\ \xi_{h_1^{0''}} \dots \xi_{h_s^{0''}} & y_1 \dots y_{N_0} \end{matrix} \middle| \overset{\circ}{\xi}_{l_1''} \dots \overset{\circ}{\xi}_{l_s''} t_1 \dots t_{P_0} \right) (*),$$

where: $r+s=2q+\sigma=h+1$, $\xi_{h'_1} < \dots < \xi_{h'_r}$; $\xi_{h_1^{0''}} < \dots < \xi_{h_s^{0''}}$ are a permutation of $\xi \xi_1 \dots \xi_h$, etc. The \circ 's mean $(\overset{\circ}{a} \overset{\circ}{b}) = (\overset{\circ}{a} b) = (a b)$; $(\overset{\circ}{a} \overset{\circ}{b}) = 0$ (same for $[]$), and kernels are defined this way.

The next step is an evaluation of the maximum infinities that may (but need not), occur, knowing the type of the singularities of (xy) and $[xy]$. Singularities are worst for *coincidence* rather than *confluence*: then (xy) is singular like ε^{-f} , $[xy]$ like ε^{-b} . Each singularity appears in a 4-fold integral, so that each integration (which we may restrict, for this purpose, to a 4-volume of dimension ε^4 around the light-cone of ξ) gives a factor ε^4 . In conclusion, divergencies for h -fold confluence occur only if

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{4h - fs - bq} \neq 0$$

(a finite value $\neq 0$ might give a logarithmic divergence) or, in terms of s and σ , if:

$$(23) \quad fs + \frac{b}{2} \sigma \leq h \left(f + \frac{b}{2} - 4 \right) + f + \frac{b}{2}.$$

In electrodynamics, $f=3$, $b=2$, so that we find that the condition for

actual occurrence of divergencies in (22) is:

$$(24) \quad 3s + \sigma \leq 4,$$

which is the standard result of all authors.

One needs therefore only consider:

$$\begin{aligned} h \text{ odd} \quad & \begin{cases} = 0, 2, 4, & s = 0, \\ = 0 & , & s = 1, \end{cases} \\ h \text{ even} \quad & \begin{cases} = 1, 3 & , & s = 1, \\ = 1 & , & s = 1, \end{cases} \end{aligned}$$

The case $\sigma=1, 3, s=1(h \text{ even})$ is immediately seen to be out, due to Furry's theorem. We also neglect here $\sigma=4, s=0$. Such terms could be easily taken care of, in any case. We retain therefore only the terms:

$$\begin{aligned} (0|0) \quad h \text{ odd:} \quad & s = 0, \quad = 0: \quad K \left(\begin{array}{c} \cdots \\ \cdots \end{array} \middle| \cdots \right), \\ (0|2) \quad & \gg \quad s = 0, \quad = 2: \quad K \left(\begin{array}{c} \cdots \\ \cdots \end{array} \middle| \begin{array}{c} \circ \circ \\ \circ \circ \end{array} \right), \\ (1|0) \quad & \gg \quad s = 1, \quad = 0: \quad K \left(\begin{array}{c} \circ \cdots \\ \circ \cdots \end{array} \middle| \cdots \right), \\ (1|1) \quad h \text{ even:} \quad & s = 1, \quad = 1 \quad K \left(\begin{array}{c} \circ \cdots \\ \circ \cdots \end{array} \middle| \begin{array}{c} \circ \cdots \\ \circ \cdots \end{array} \right). \end{aligned}$$

After some heavy combinatorics, we find again that some magic is at work.

First, define some new, totally divergent « kernels », or better « cores of the divergencies »:

$$(25) \quad L \left(\begin{array}{c} \circ \hat{x}_1 \cdots \hat{x}_N \hat{\xi} \\ \circ \hat{y}_1 \cdots \hat{y}_N \hat{\xi} \end{array} \middle| \begin{array}{c} \hat{\xi} \hat{t}_1 \cdots \hat{t}_P \end{array} \right) = \sum_{N(P+1)} \frac{\lambda^N}{N!} d_{\hat{\xi}}^1 \cdots d_{\hat{\xi}}^N \gamma^1 \cdots \gamma^N \cdot \left(\begin{array}{c} \circ \hat{x}_1 \cdots \hat{x}_N \hat{\xi} \hat{\xi}_1 \cdots \hat{\xi}_N \\ \circ \hat{y}_1 \cdots \hat{y}_N \hat{\xi} \hat{\xi}_1 \cdots \hat{\xi}_N \end{array} \middle| \begin{array}{c} \hat{t}_1 \cdots \hat{t}_P \hat{\xi} \hat{\xi}_1 \cdots \hat{\xi}_N \end{array} \right).$$

Then, use (25):

$$(0|0) \text{ leads only to } \int d\xi \gamma^\xi L \left(\begin{array}{c} \xi \\ \xi \end{array} \middle| \xi \right), \quad (1|0) \text{ leads to } \int d\xi L \left(\begin{array}{c} \hat{x} \hat{\xi} \\ \hat{y} \hat{\xi} \end{array} \middle| \xi \right), \text{ etc.}$$

Finally, one finds that *necessary and sufficient conditions* for the prescription

adopted to be a renormalization are that:

$$(26) \quad \left\{ \begin{aligned} \int d\xi L \left(\begin{smallmatrix} \hat{x} & \xi \\ \hat{y} & \xi \end{smallmatrix} \middle| \xi \right) &= \alpha(\lambda) \cdot (xy) + \beta(\lambda) \frac{\partial(xy)}{\partial m_f}, \\ \int d\xi L \left(\begin{smallmatrix} \xi \\ \xi \end{smallmatrix} \middle| \xi \hat{x} \hat{y} \right) &= \gamma(\lambda) \cdot [xy] + \delta(\lambda) \frac{\partial[xy]}{\partial(m_b^2)}, \\ \int d\xi L \left(\begin{smallmatrix} \hat{x} & \xi \\ \hat{y} & \xi \end{smallmatrix} \middle| \xi \hat{t} \right) &= \varepsilon(\lambda) \int d\tau (x\tau) \gamma^r(\tau y) [\tau t]. \end{aligned} \right.$$

Calling

$$a_0(\lambda) = \int d\xi L \left(\begin{smallmatrix} \xi \\ \xi \end{smallmatrix} \middle| \xi \right),$$

one has then, after some more work:

$$(27) \quad [1 - \lambda\alpha(\lambda) - \tfrac{1}{2}\lambda\gamma(\lambda) - \varepsilon(\lambda)] \frac{\partial K_{N_0, P_0}}{\partial \lambda} - [a_0(\lambda) + N_0\alpha(\lambda) + \tfrac{1}{2}P_0\gamma(\lambda)]K_{N_0, P_0} + \\ - \beta(\lambda) \frac{\partial}{\partial m_f} K_{N_0, P_0} - \delta(\lambda) \frac{\partial}{\partial(m_b^2)} K_{N_0, P_0} = \int d\xi \gamma^\xi K \left(\begin{smallmatrix} x_1 \dots x_{N_0} & \xi \\ y_1 \dots y_{N_0} & \xi \end{smallmatrix} \middle| \xi \ t_1 \dots t_{P_0} \right).$$

At r.h.s. of (27), the same step has been accomplished, that changed, in the example, $\sum K'$ into $\sum' K$ (here: $\int_1 d\xi \rightarrow \int d\xi$).

It is at this point very easy to see that, if we consider λ, m_f, m_b as functions of new parameters $\bar{\lambda}, \bar{m}_f, \bar{m}_b$:

$$(28) \quad \left\{ \begin{aligned} \lambda &= \lambda(\bar{\lambda}), \\ m_f &= \bar{m}_f - \delta m_f(\bar{\lambda}) \\ m_b^2 &= \bar{m}_b^2 - \delta m_b^2(\bar{\lambda}), \end{aligned} \right.$$

and set:

$$(29) \quad K_{N_0, P_0}(\lambda, m_f, m_b) = A Z_2^{N_0} Z_3^{P_0} \hat{K}(\bar{\lambda}, \bar{m}_f, \bar{m}_b), \quad \lambda = \lambda(\bar{\lambda}) = Z_1 Z_2^{-1} Z_3^{-\frac{1}{2}} \bar{\lambda},$$

we have

$$(30) \quad \frac{\partial \hat{K}_{N_0, P_0}(\bar{\lambda}, \bar{m}_f, \bar{m}_b)}{\partial \bar{\lambda}} = \int d\xi \hat{K}_{N_0+1, P_0+1} \left(\begin{smallmatrix} \dots & \xi \\ \dots & \xi \end{smallmatrix} \middle| \xi \dots \right),$$

provided $A, Z_1 Z_2 Z_3, \lambda(\bar{\lambda})$, satisfy equations in $\bar{\lambda}$ which are easily written and

solved, formally, in terms of $\alpha(\lambda)$, ..., $\varepsilon(\lambda)$. Thus:

$$\bar{\lambda}(\lambda) = \int_0^\lambda \frac{d\lambda}{f(\lambda) \left[1 - \int_0^\lambda \frac{1 - f(\lambda) - \varepsilon(\lambda)}{\lambda f^2(\lambda)} d\lambda \right]} \quad \text{with} \quad f(\lambda) = 1 - \lambda\alpha(\lambda) - \frac{\lambda}{2}\gamma(\lambda) - \varepsilon(\lambda).$$

This shows that the terms dropped can be identified with (infinite, in a theory with divergencies) standard renormalizations.

3.5. Independence from perturbative expansions.

a) First, the argument used for the example shows that (30), used to generate the perturbative expansions, yields everything renormalized.

We conclude that what we have done shows simply that if one changes $\int d\xi_1 \dots \int d\xi_N$ into $\int d\xi_1 \dots \int d\xi_N$ renormalization obtains automatically.

To prove that this fact has nothing to do with perturbation theory, one has simply to start with (30), in which $\int d\xi$ is changed into $\int d\xi$: iterations lead, in virtue of the branching equations, immediately back to (27). The same is true in all possible branching equations. Anything that the change $\int d\xi \rightarrow \int d\xi$ adds is of the form that counterterms would produce.

The renormalized theory obtains from the unrenormalized one by changing everywhere $\int d\xi$ with $\int d\xi$.

b) Eq.'s (26), in a covariant theory, are a simple consequence of the Lorentz-invariant character of the definition of $\int d\xi$.

The proof is of the standard type, and need not be reported here.

c) Provided one makes sure that a prescription satisfies (26), it can be applied to any non-covariant theory, or to such things as Bethe-Salpeter's equations, etc.

d) We have essentially shown how *regularization* must be handled so that it becomes renormalization. For the so-called non-renormalizable theories, final results may depend upon the prescription adopted: we have only a regularization procedure. One may ask whether the condition that the resulting theory be unitary, or some such thing, may suffice to render the prescription unambiguous: if it were so, all theories, or some of them at least, would become «renormalizable». This is a bare chance, please do not take it as a conjecture.

e) In conclusion, the theory is now on a well defined mathematical basis: things make sense, existence can be investigated. From a «practical» point of view, I believe that many computations should become far simpler with these new techniques than they have been in the past.

Investigation into Interference Phenomena at Extremely low Light Intensities by Means of a Large Michelson Interferometer.

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(ricevuto il 18 Aprile 1958)

CONTENTS. — 1. Introduction. — 2 The instrument. — 3. The results of measurements.

1. — Introduction.

According to the wave theory of light the relative intensity distribution inside an interference or diffraction pattern is independent of the intensity of the light source, which produces the pattern.

When, at the beginning of the century the corpuscular nature of light was reintroduced in its modern version, some doubts arose, whether interference patterns for low intensities are still the same as for higher intensities.

Among others J. J. THOMSON (see preface of G. I. TAYLOR's paper) put the question whether a diffraction image produced by an extremely low intensity of light showed the same relative distribution of light as a pattern produced by a source of normal intensity. On the suggestion of J. J. THOMSON, G. I. TAYLOR ⁽¹⁾ carried out experiments on diffraction patterns produced by different intensities of light and came to conclude that the interference phenomena are independent of absolute intensity. Later A. J. DEMPSTER and H. F. BATHO ⁽²⁾ carried out experiments in somewhat greater detail, using an echelon and they also came to the conclusion that the intensity pattern does not depend on the intensity of the light producing the pattern.

⁽¹⁾ G. I. TAYLOR: *Proc. Camb. Phil. Soc.*, **15**, 114 (1909).

⁽²⁾ A. J. DEMPSTER and H. F. BATHO: *Phys. Rev.*, **30**, 644 (1927).

It must be emphasized that from quantum theory it follows that intensity patterns do not depend on intensity of light and there are no difficulties in interpreting the interference phenomena at arbitrary low intensities as long as we are satisfied with a purely mathematical description of the phenomena and do not try to picture in detail the process. If we try, however, to do so we come to paradoxical conclusions, as was discussed, *e.g.* by BORN ⁽³⁾, JORDAN ⁽⁴⁾, DIRAC ⁽⁵⁾ and many others.

Regarding these paradoxes many physicists accept the point of view that it is sufficient to give a mathematically consistent picture of micro-phenomena and that it is unnecessary to try to give a picture of what is happening in detail. Other physicists, *e.g.* SCHRÖDINGER ⁽⁶⁾, regard this state of affairs as unsatisfactory and take the impossibility of interpreting the mathematical formalism as the «sign of a crisis».

It is seen that on the theoretical side there is no complete agreement as to the interpretation of the wave-particle phenomenon. Furthermore, a detailed scrutiny of the experiments of G. I. TAYLOR and of DEMPSTER and BATHO has shown that the results of these experiments are less convincing than usually assumed ⁽⁷⁾, therefore it seemed to be worth-while to carry out new experiments and to investigate the question anew. Such new experiments seemed the more worthwhile since making use of modern methods of counting single photons, the experiments could be carried out in a manner far more direct than was possible formerly.

Using the photon counting technique we investigated the interference phenomena with a Michelson interferometer of an arm length of 10 cm ⁽⁸⁾. The results of this experiment showed that there is no difference between the intensity distributions of the high and low intensity interference patterns. In the following section we should like to make a few additional remarks on the problem of low intensity interference phenomena.

A photon might be assumed to be a packet of length comparable with the coherence length of a spectral line. If the dimensions of the (*e.g.* Michelson type) interferometer are much smaller than the «size» of the photon, then the interference phenomenon might be accounted for by picturing that the photon floods the whole of the arrangement simultaneously and thus comes under the influence of the whole of the optical arrangement and, in particular, under the influence of the two mirrors at the ends of the arms. According to

⁽³⁾ M. BORN: *Optik* (Berlin, 1933), p. 465.

⁽⁴⁾ P. JORDAN: *Anschauliche Quantentheorie* (Berlin, 1936), p. 14.

⁽⁵⁾ P. A. M. DIRAC: *Principles of Quantum Mechanics* (Oxford, 1955), p. 7.

⁽⁶⁾ E. SCHRÖDINGER: *Rencontres Internationales de Genève* (1952).

⁽⁷⁾ L. JÁNOSSY and Zs. NÁRAY: *Magy. Fiz. F.*, **6**, 105 (1958), (in Hungarian).

⁽⁸⁾ L. JÁNOSSY and Zs. NÁRAY: *Acta Phys. Hung.*, **7**, 463 (1957).

the measurements of LUMMER and GEHRKE⁽⁹⁾ the coherence length (*e.g.* in the case of the green mercury line) can be assumed to be of the order of 1 m. This length is indeed larger than the dimensions of the interferometer used in our first experiment⁽⁸⁾. For this reason we decided to repeat the interference experiment with an interferometer with arms long compared with the coherence length; thus we have carried out measurements with an interferometer with about 14 m long arms. Such an interferometer is appreciably larger than the coherence length of the photons, therefore, regarding it from the classical point of view, a wave packet about 1 m long, when falling on the optical splitting device (*e.g.* on the semi-transparent mirror of the Michelson interferometer) has to break up into two parts, each travelling along one of the arms; after reflection these two parts unite again. The difference of path lengths along the two arms is of course small and thus the two parts of the wave packet when considered classically unite in corresponding phase and give rise to interference.

2. - The instrument.

In carrying out the experiments it was necessary to draw a limit between what may be called a small intensity and what may be called a large intensity. No such distinction can be made on the basis of either quantum mechanics or pure classical physics as according to both of them the phenomena are strictly linear. However, from a primitive corpuscular aspect of light we may interpret as small intensity one for which on the average less than one photon is contained at any time inside the apparatus. Alternatively, if we picture the photons as light impulses containing many wavelengths and being of the length of the optical coherence length, then we may denote an intensity as small provided the lengths of the single wave packets about equal the coherence length and do not appreciably overlap.

Although these criteria for small intensities have no particular significance according to the mathematical formalism of the accepted theories, nevertheless, they give a limit such that for intensities below that limit the paradoxes of interpretation discussed in the preceding article⁽⁷⁾ arise.

The optical path l' of a photon in our interferometer is of the order of twice the arm length, l , *i.e.* in our case where $l = 13.85$ m

$$l' \sim 2l \sim 30 \text{ m.}$$

Thus the condition for only a few photons to be contained simultaneously

⁽⁹⁾ G. LUMMER and N. GEHRKE: *Verh. Deutsch. Phys. Ges.*, **4**, 337 (1902).

inside the arrangement is that the intensity N expressed in photons per second should be

$$N < N_0 = \frac{c}{\bar{\nu}} \sim \frac{3 \cdot 10^{10}}{3 \cdot 10^3} = 10^7 \text{ photons/s.}$$

Taking the second criterion, *i.e.* that the «photon bands» should not appreciably overlap we may require for a small intensity N that

$$N < N_0 = \frac{c}{\Lambda} \sim 3 \cdot 10^8 \text{ photons/s,}$$

where Λ is the coherence length, the order of magnitude of which we may assume to be ~ 1 m.

We carried out measurements with a Michelson interferometer with $l = 13.85$ m arm length counting the photons in the pattern IP by means of photomultiplier P (Fig. 1). The interference pattern consisted of vertical fringes. The fringes were projected through a vertical slit onto the cathode of the photomultiplier. The multiplier together with the slit was moved by small steps along a horizontal path and thus the rate of incident photons was counted in various points of the fringe system. In this way we determined the intensity distribution in the interference pattern both for high and low overall intensity. The experimental procedure and the evaluation of the results were done in a way similar to those described in a former paper dealing with the small interferometer (⁸).

So as to get reliable results great precautions had to be taken. In order to measure out an interference pattern point by point a measuring time of about an hour was required. During this period the pattern must not move by any appreciable fraction of the fringe width as otherwise the patterns obtained for high and low intensities cannot be compared quantitatively. At the same time the low inten-

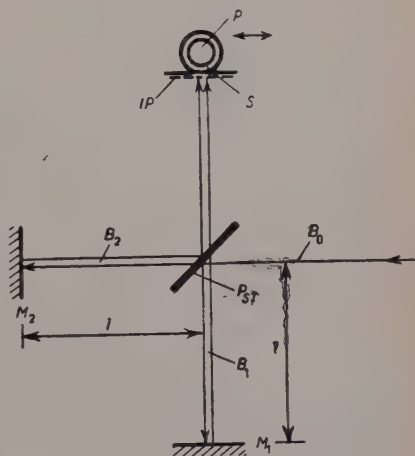


Fig. 1. — Arrangement of the measuring set. The incoming beam B_0 is split by the semi-transparent mirror P_{ST} into beams B_1 and B_2 . The beams B_1 and B_2 reflected by mirrors M_1 and M_2 , respectively, form in the plane of a slit S of the multiplier P the interference pattern IP . This interference pattern is counted out step-by-step by moving the multiplier P . The arm length of the interferometer was $l = 13.85$ m.

sity pattern is of such low intensity as to be practically invisible and therefore no check can be made while the measurement is in progress as to the constancy of the pattern. We note that so as to avoid a fringe shift of $1/10$ of a fringe the arms must remain constant to an amount of the order of $\lambda/20$. A lengthening of $\lambda/20$ of an arm might be caused by a temperature change of about 10^{-5} °C.

So as to achieve the necessary stability the apparatus was built in an underground tunnel cut in rock 30 m under the surface of the earth. The length of the tunnel was 20 m and its width 2.5 m. The interferometer was mounted directly on the concrete floor of the tunnel.

The tunnel appeared to be sufficiently stable both regarding temperature and mechanical disturbances so as to be suitable for the measurements. However, the manipulations with the arrangement disturbed the interferometer to such an extent that it was found necessary to operate the arrangement entirely by remote control from a different compartment. It was found that after all necessary adjustments had been carried out it was necessary to leave the arrangement for a few days to rest. After such a period all mechanical tensions relaxed and temperature equilibrium was obtained and using the fully automatized gadgets successful series of measurements could be carried out. The automatics is described in greater detail elsewhere ⁽⁸⁾.

Because of the great sensitivity of the interferometer to any changes in the temperature the dark current of the multipliers could not be reduced by cooling. Cooling would have been of disadvantage among other reasons also because it reduces the sensitivity of the multipliers ⁽¹⁰⁾. Particular precautions had to be taken so as to reduce the dark current pulses of the multipliers without reducing the sensitivity. The dark current pulses of the multipliers could be reduced without cooling by methods described elsewhere ⁽¹¹⁻¹³⁾.

While the arrangement was locked up in the tunnel the pattern could be checked from outside by help of a periscope inserted by means of a mechanical manipulator. With help of the manipulator the intensity could be increased to such an extent that visual observation or photographic recording was rendered possible. If the interference pattern seen through the periscope showed any change the mirrors could be slightly adjusted by means of another manipulator so as to bring the pattern back to normal. From time to time we also took photographs of the pattern seen through the periscope.

The block diagram of the arrangement is shown in Fig. 2. The light is obtained from a high-pressure mercury discharge tube *L*. The light passes

⁽¹⁰⁾ ZS. NÁRAY: *Ann. d. Phys.* **20**, 386 (1957).

⁽¹¹⁾ ZS. NÁRAY: *Acta Phys. Hung.*, **5**, 159 (1955).

⁽¹²⁾ ZS. NÁRAY and P. VARGA: *Brit. Journ. of Appl. Phys.*, **8**, 377 (1957).

⁽¹³⁾ ZS. NÁRAY: *Journ. Sci. Instr.*, **33**, 476 (1956).

through an optical gadget which can be used to reduce the light intensity by factors equal to powers of ten ⁽¹⁴⁾. The reduced light beam is passed through the monochromator M , which selects the line $\lambda = 5461 \text{ \AA}$. The monochromatic beam is focused by means of the lens L_0 on to the slit S_0 of the interferometer.

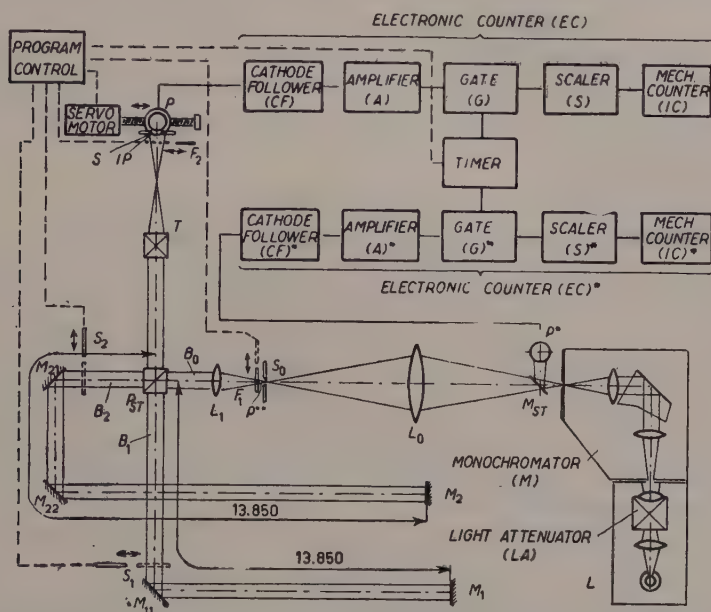


Fig. 2. — Block diagram of the experimental arrangement. For a description see Sect. 2.

A second lens L_1 produces a nearly parallel beam B_0 ; which falls on the splitting prism P_{ST} and is divided into two coherent components B_1 and B_2 . As we had not enough room to let both beams B_1 and B_2 travel a distance of 14 m in one direction we turned the reflected beam by means of a mirror M_{11} through an angle of 90° and let it move backwards along the direction of the incoming beam; similarly we turned the transmitted beam by means of two mirrors M_{21} and M_{22} by twice 90° and thus both beams B_1 and B_2 traveled most of their paths in the direction of the tunnel into which the whole arrangement was mounted. The beams B_1 and B_2 are returned by the mirrors M_1 and M_2 . Thus the beams return to the prism P_{ST} . The transmitted part of B_1 and the reflected part of B_2 pass through the telescope T and produce an interference pattern IP in the plane of the slit S of the photomultiplier. The photomultiplier together with the slit could be moved in a direction per-

⁽¹⁴⁾ L. JÁNOSSY and Zs. NÁRAY: *Acta Phys. Hung.*, **5**, 153 (1955).

pendicular to the direction of the fringes and thus the intensity distribution could be determined by counting the rates of photons in various parts of the pattern.

In the course of the measurements a beam of about 10^{10} photons/s falling on the slit S_0 represented the large intensity. The absolute measurement of this intensity was performed by a thermopile⁽¹⁵⁾. To obtain the small intensity we placed an optical filter F_1 close behind the slit S_0 , which reduced the intensity by a factor 10^4 , thus the small intensity amounted to about 10^4 photons/s, the latter intensity can be regarded as small taking either of the criteria of Sect. 2.

When measuring with the large intensity it was found convenient to place filter F_2 close before the slit S of the multiplier. The filter F_2 reduced the light intensity, similar to the filter F_1 , by a factor 10^4 . The introduction of the filter F_2 was found convenient as by means of this filter the large intensity—after the coherent beams had been mixed—could be reduced to the same intensity as is obtained from the small intensity beam falling into the interferometer. This reduction of the strong beam just before entering the multiplier is irrelevant from the point of view of the principle of the experiment. Indeed, if the filter F_1 is placed before the input of the interferometer, then *weak beams* of light interfere and produce a pattern. If, on the other hand, the filter F_2 is introduced before the slit we have strong beams of light interfering and producing a pattern, the filter F_2 merely not allowing all the photons of the pattern to pass on to the cathode of the multiplier.

The advantage of using the filter F_2 is, that in this way the multiplier is exposed both in case of large and small input intensities to approximately equal intensities of light. Thus overloading of the multiplier is avoided and a stable performance rendered possible.

The interference pattern obtained covered a disc of 5 mm diameter. We used a square area of about 2.0 mm side length, one side being parallel to the slit and so to the fringes. The slit was 0.2 mm wide and was moved in steps of 0.25 mm each across the pattern. We counted in each position for 10 s.

Denote by x the distance of the slit from the vertical symmetry axis of the square and $\nu'(x)$ the number of photons counted if the interference pattern is produced by low intensity with the slit in position x during 10 s. Similarly, denote by $\nu''(x)$ the number of photons counted under similar conditions for high intensity interference. As in the latter case the light is reduced, after interference has taken place, by the filter F_2 , the numbers $\nu'(x)$ and $\nu''(x)$ are expected to be about equal.

(15) ZS. NÁRAY and M. TÓTH: *Magy. Fiz. F.* **5**, 7 (1957) (in Hungarian).

So as to be able to compare the two intensity distributions the background count for each individual reading had to be determined. This was done by introducing shutters S_1 and S_2 (see Fig. 2) into the beams B_1 and B_2 after each reading. The count thus obtained, with the beams of the interferometer blocked, were due to stray light and to the dark current pulses of the multiplier. Denoting by $\nu'_0(x)$ and $\nu''_0(x)$ the counts obtained under conditions similar to those under which $\nu'(x)$ and $\nu''(x)$ were obtained, except for the shutters S_1 and S_2 being closed during the measurements of the former, we have for the counts caused by the interference pattern itself

$$p'(x) = \nu'(x) - \nu'_0(x), \quad p''(x) = \nu''(x) - \nu''_0(x)$$

for low and high intensities, respectively.

So as to compare the intensity distributions with high and low input intensities, one might consider the ratio

$$(1) \quad A(x) = \frac{p'(x)}{p''(x)}.$$

The theoretical prediction is that

$$A(x) = \text{const.}$$

If we assume the filters F_1 and F_2 to be exactly equal, we expect

$$(2) \quad A(x) = 1.$$

However, the relation (2) is not very suitable for a precise checking of the theory. Because of the optical faults of the system the functions $p'(x)$ and $p''(x)$ are not exactly sinusoidal. Computing thus $A(x)$ from (1) we have roughly speaking to judge whether

$$A(x_{\max}) \text{ equal or not equal } A(x_{\min}),$$

where x_{\max} and x_{\min} are the co-ordinates of the interference maximum and minimum. However, the determination of $A(x_{\min})$ is necessarily inaccurate, firstly because the counts $p'(x_{\min})$ and $p''(x_{\min})$ are in any case small, secondly because the minima of the curves are not very well defined and are subject to even greater relative fluctuations caused by outside disturbances than the points of higher intensity. So as to get quantities which are more suitable for such a comparison, we have inserted after each period of measurement of $\nu'(x)$ and $\nu'_0(x)$ a measurement with one of the shutters, *e.g.* S_1 open and

the shutter S_2 closed. Denoting the result of the measurement by $v'_1(x)$ and similarly by $v''_1(x)$ the corresponding measurement at conditions of high intensity, then

$$\bar{p}'(x) = v'_1(x) - v'_0(x) \quad \text{and} \quad \bar{p}''(x) = v''_1(x) - v''_0(x)$$

are the light intensities reaching the multiplier along one arm of the interferometer with the other arm blocked. Thus $2\bar{p}'(x)$ and $2\bar{p}''(x)$ represent the average intensities of the interference field; *i.e.* the intensities which would be obtained if the two beams were to add up their intensities without giving rise to interference. The differences

$$p'(x) - 2\bar{p}'(x) \quad \text{and} \quad p''(x) - 2\bar{p}''(x)$$

give thus the effect of interference on the intensity in the point x . In particular the ratios

$$i'(x) = \frac{p'(x) - 2\bar{p}'(x)}{2\bar{p}'(x)} \quad \text{and} \quad i''(x) = \frac{p''(x) - 2\bar{p}''(x)}{2\bar{p}''(x)},$$

give the relative intensity variation inside the pattern. If the interference pattern would fade out at low intensities as could be anticipated in terms of some model, we would expect $i'(x)$ to differ from $i''(x)$; *i.e.* $i'(x)$ would show a variation less pronounced than $i''(x)$, and if the pattern would fade out altogether at a very low intensity $i'(x)$ would approach a constant at that low intensity. On the contrary if the pattern does not depend on intensity $i'(x)$ and $i''(x)$ do not differ. The comparison of $i'(x)$ and $i''(x)$ is advantageous as these values should be expected to be equal regardless of whether or not the filters F_1 and F_2 are exactly equal; furthermore, these values at the intensity minimum and maximum are statistically about equally well defined, thus the comparison of $i'(x)$ and $i''(x)$ for $x = x_{\min}$ and $x = x_{\max}$ gives a statistically good criterion as to whether the interference pattern is not independent of intensity.

During the series of measurements we had to carry out constant checks as to the stability of the arrangement. Among others, we introduced a mirror M_{st} into the beam before it entered into the interferometer (Fig. 2) splitting off by means of this mirror a constant fraction of the intensity. The intensities of this split-off beam were constantly recorded by a multiplier P^* . We rejected all those readings where the overall intensity indicated by P^* differed from an average value.

The photomultipliers were fed from a stabilized high tension set constructed specially for the purpose ⁽¹⁶⁾. Furthermore, we took care to keep the multipliers all the time in darkness. Details can be found in other publications ⁽¹⁰⁻¹³⁾.

3. - The results of measurements.

We carried out 12 runs of measurements and thus obtained 12 sets of determinations of the relative intensity variations $i'(x)$ and $i''(x)$. The results of one of the series are shown in Table I and Fig. 3. We note that the average

TABLE I.

x	$\nu'(x)$	$\nu''(x)$	$\nu'_1(x)$	$\nu''_1(x)$	$\nu'_0(x)$	$\nu''_0(x)$
-1.00	6 550	13 350	5 050	4 850	1 600	1 600
-0.75	8 200	13 900	4 700	4 850	1 600	1 600
-0.5	15 050	10 150	4 950	4 600	1 550	1 600
-0.25	15 100	13 300	4 950	4 850	1 600	1 550
0	7 000	7 000	5 000	4 800	1 600	1 550
0.25	1 800	2 250	4 750	4 600	1 550	1 550
0.5	6 000	7 250	4 800	4 400	1 600	1 550
0.75	12 550	6 000	4 550	4 000	1 550	1 650
+1.00	13 400	9 400	3 750	3 750	1 500	1 600

intensity $2\bar{p}(x)$ was of the order of $6 \cdot 10^4$ photons/s, while $\nu_0(x)$ was of the order of $1,6 \cdot 10^4$ photons/s.

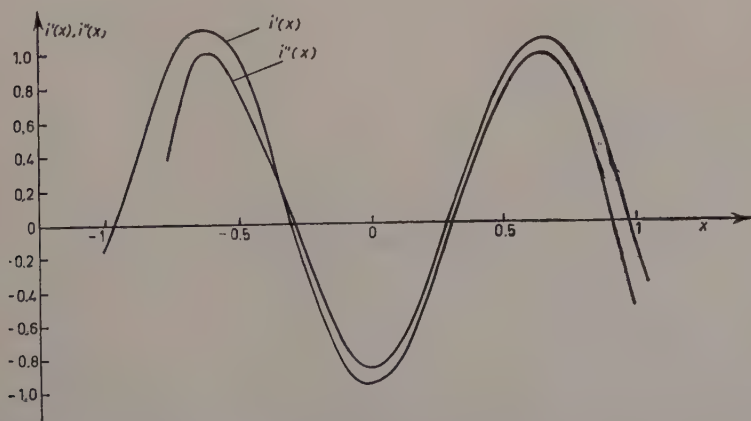


Fig. 3. - Relative intensity variation of the interference pattern in a typical measuring run.

⁽¹³⁾ Zs. NÁRAY and K. ZSDÁNSZKY: *Žurnal Priborii i Technika Erperimenta*, 108 (1956).

The curves $i'(x)$ and $i''(x)$ do not show any systematic deviations. An exact analysis as to the largest deviations between $i'(x)$ and $i''(x)$ which would be still compatible with the errors of measurement could not be carried out for the present measurements. It seems, however, that any systematic deviation greater than $(10 \div 20) \%$ is unlikely.

The quantitative analysis of the possible deviations between $i'(x)$ and $i''(x)$ would require rather more material than we had obtained, when owing to outside circumstances the measurements had to be interrupted. We did not think it worth-while to restart the measurements at a later period as the results as they are confirm very well, even if only qualitatively, the expectation of theory for such small intensities of light where the paradoxes described in another article hold. As the result of our measurements we have therefore to conclude that *interference phenomena are perfectly normal even at such low intensities where at one time in average less than one photon is to be found in the arrangement; this is true even if the dimensions of the arrangement greatly exceed the coherence length of the photons giving rise to the patterns.*

* * *

We express our thanks to Mssr. K. TITSCHKA, J. KMETYÓ and J. FÜRJES for their help in building the arrangements, further to Miss M. TASSALY and Miss A. FODOR for their help in evaluating the photographic records.

On the Causal Character in Field Theory (*).

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(ricevuto il 23 Aprile 1958)

CONTENTS: — 1. Introduction. — 2. Causal function. — 3. The causal character of the vacuum- T -function. — 4. The causal character of the S -matrix. — 5. Causal ambiguity and renormalization constants. — 6. Causality in the dispersion relation. — 7. Concluding discussion. — APPENDIX I: Non-local interaction. — APPENDIX II: A note on the Lagrangian including higher derivatives.

1. — Introduction.

We usually expand the field variable in terms of the suitable orthonormalized complete set, and quantize every mode as independent harmonic oscillator. This method has its base on the formal correspondence to particle dynamics, and all modified methods of quantization ⁽¹⁾ are proved to be equivalent to this original one. Field theory is now very tightly formulated in the so-called Lagrangian formalism, and, at first sight, there seems to be no room to modify the theory or to introduce new concepts into the scheme. The Lagrangian formalism is excellent in the point that it automatically warrants the internal consistence between wave equations, commutation relations, Schrödinger equation, invariance and conservation laws, etc. ⁽²⁾. On the other

(*) The text of this article is translated from the *Soryūshiron-kenkyū* (mimeographed circular in Japanese), **14**, 599 (1957).

⁽¹⁾ There are many modified methods, but we cite only the recent works. R. P. FEYNMAN: *Phys. Rev.*, **76**, 749 (1949); P. T. MATTHEWS and A. SALAM: *Nuovo Cimento*, **1**, 120 (1955); R. E. PEIERLS: *Proc. Roy. Soc.*, **214**, 143 (1952); M. CINI: *Nuovo Cimento*, **9**, 501 (1952); F. COESTER: *Phys. Rev.*, **95**, 1318 (1954).

⁽²⁾ See, for example, Y. NAMBU: *Prog. Theor. Phys.*, **7**, 131 (1952); J. SCHWINGER: *Phys. Rev.*, **82**, 914 (1951).

hand, it is necessary to improve the theory in any sense, because we are worried about the appearance of divergence difficulties in the present formalism. The renormalization method is limited not only in practical uses, but also in formal aspects ⁽³⁾. The theory of relativistic cut-off had been investigated by many authors ⁽⁴⁾, but we could not find any consistent method.

The improvement, which is required at this stage of field theory, seems not to be the merely formal modification of the formalism or the more precise treatment of mathematical features, but it looks better to introduce any new physical image. For this purpose, it will be necessary to understand the structure of field theory as physically and intuitively as possible, before we consider what should be introduced. The investigation from such a point of view will make clear the physical significance of any modification of some parts of the scheme, and what kind of physical concepts can be introduced. It will serve also in order to get the foresight about the general situation of future problems and to understand the limit of some standpoint.

Field theory is now arranged in a beautiful form, but, just for this reason, it becomes rather difficult to understand the physical image included in the formalism. It seems more fruitful to analyse the structure of field theory not from the formal aspects, but from the point of view of physical correspondence. In this article, we investigate the structure of field theory on the ground of the concept of causality, that was settled by STÜCKELBERG ⁽⁵⁾. STÜCKELBERG and his coworkers attempted to formulate the propagation of classical waves in the quantum theoretical mode, and, there, the formal correspondence to the quantum mechanics of particles was not used explicitly. This formulation is very similar to the Feynman's relativistic formulation of field theory ⁽¹⁾, but their standpoint has merits, which are not included in the latter.

We can understand the situation that the fundamental assumptions to construct S -matrix are Lorentz invariance, unitarity and causality, if we take into account the causality in the sense of STÜCKELBERG. It becomes clear what the causality requirement imposes on the physical system, so we can get an intuitive image on the structure of S -matrix. Further, it was disclosed that the so-called renormalization procedure could be connected with the causal ambiguity, which was due to the insufficiency of the physical correspondence. It is very interesting that there remains such an arbitrariness yet, even though the present field theory seems to be constructed very tightly.

⁽³⁾ T. D. LEE: *Phys. Rev.*, **95**, 1329 (1954); L. D. LANDAU: *Niels Bohr and the Development of Physics* (London, 1955), p. 52; G. KÄLLÉN and W. PAULI: *Dan. Mat. Fys. Medd.*, **30**, No. 7 (1955).

⁽⁴⁾ C. BLOCH: *Dan. Mat. Fys. Medd.*, **27**, No. 8 (1952); C. HAYASHI: *Prog. Theor. Phys.*, **10**, 533 (1953); W. PAULI: *Nuovo Cimento*, **10**, 648 (1953); E. C. G. STÜCKELBERG and G. WANDERS: *Helv. Phys. Acta*, **27**, 667 (1954).

⁽⁵⁾ E. C. G. STÜCKELBERG and D. RIVIER: *Helv. Phys. Acta*, **23**, 215 (1950).

Of course, nothing is actively extracted from this arbitrariness, about what we must introduce. But, we can take out a standpoint to make the new physical image that fills up the insufficiency of the correspondence.

We start our survey from the explanation of Stückelberg's correspondence-theoretical interpretation of causality, and of Fierz's interpretation from the standpoint of the corpuscular aspect (Sect. 2). In Sect. 3, we analyse the causal character of the so-called vacuum- T -function, and, there, it will be studied what the causality requirement imposes on the physical system. The causal character of S -matrix will be made clear in Sect. 5. The insufficiency of the causal correspondence to the classical wave image, and the relation between the causal ambiguity and the renormalization procedure will be taken up in Sect. 6. In Sect. 8, we propose a direction to modify the field theory, in accordance with our analysis. The connection between Stückelberg's causality and the so-called microcausality also will be explained there. The causal aspect used in the dispersion relation will be discussed in Sect. 7. Appendices are devoted to the discussions about the limit of the theory of non-local interaction, from the standpoint of S -matrix formalism.

2. - Causal function.

STÜCKELBERG⁽⁵⁾ expresses the causality in field theory as the fact that *action is carried by a wave of positive frequency which propagates from past to future*. This is a correspondence from the standpoint of wave aspect.

We consider a scalar field, as an example, to interpret how we can formulate this idea. We define a Δ^+ -function constructed by an ortho-normalized complete set (*) $\{u_\alpha\}$ in the following way,

$$(2.1) \quad i \Delta^+(x, y) \equiv \sum_{\alpha} u_{\alpha}(x) u_{\alpha}^*(y).$$

This Δ^+ -function has the property,

$$(2.2) \quad \int d^3y \Delta^+(x, y) \frac{\overleftrightarrow{\partial}}{\partial y_0} u_{\beta}(y) = u_{\beta}(x),$$

(*) This set needs not to be the plane wave. The ortho-normalization relation is given by

$$-i \int d^3x u_{\alpha}(x) \frac{\overleftrightarrow{\partial}}{\partial x_0} u_{\beta}^*(x) = \delta_{\alpha\beta},$$

here

$$f(x) \frac{\overleftrightarrow{\partial}}{\partial x_0} g(x) = f(x) \frac{\partial g}{\partial x_0} - \frac{\partial f}{\partial x_0} g(x).$$

for the arbitrary function u_β belonging to the set $\{u_x\}$. Namely, if we have a wave with positive frequency at the time y_0 , the $\Delta^+(x, y)$ -function brings this wave to the time x_0 without altering the configuration. So, we can take $\Delta^+(x, y)$ as the causal propagator $\Delta^c(x, y)$, if $x_0 > y_0$. $\Delta^+(x, y)$ is equal to its retarded part, $\Delta_{\text{ret}}^+(x, y)$, there. The corresponding propagation function is $D_{\text{ret}}(x, y)$ in the classical Maxwell solution, and the quantum theoretical feature is here introduced by taking only the positive frequency part of it. D_{ret} is not zero only on the future light cone, but D_{ret}^+ has its non-zero value near the light cone too.

Next, we must extend the definition of this $\Delta^c(x, y)$ function to the case $y_0 < x_0$, without disturbing relativistic covariance. For this purpose, we take into account the fact that, if $x_0 > y_0$, $\Delta^+(x, y)$ function can be expressed as

$$(2.3) \quad \Delta^+(x, y) \approx \frac{i}{2} \{ \Delta_1(x, y) - i \Delta(x, y) \} \quad (x_0 > y_0),$$

by the two independent (symmetric and antisymmetric) solutions of the homogeneous wave equation,

$$\begin{aligned} \Delta_1 &= -i(\Delta_{\text{ret}}^+ - \Delta_{\text{av}}^+ - \Delta_{\text{ret}}^- + \Delta_{\text{av}}^-), \\ \Delta &= \Delta_{\text{ret}}^+ - \Delta_{\text{av}}^+ + \Delta_{\text{ret}}^- - \Delta_{\text{av}}^-. \end{aligned}$$

Here, the symbol \approx means that the left hand side of this equation is equal to the right hand side under the condition $x_0 > y_0$. We here assume that $\Delta^c(x, y)$ always can be expressed by a linear combination of Δ_1 - and Δ -function, that is

$$\begin{aligned} \Delta^c(x, y) &= \frac{i}{2} \{ \Delta_1(x, y) + a \Delta(x, y) \} = \\ &= \frac{i}{2} \{ (-i + a) \Delta_{\text{ret}}^+ + (i - a) \Delta_{\text{av}}^+ + (i + a) \Delta_{\text{ret}}^- - (i + a) \Delta_{\text{av}}^- \}. \end{aligned}$$

If $x_0 > y_0$, it must be $a = -i$, as was shown above. In the case $x_0 < y_0$, the effective parts are Δ_{av}^+ . If Δ_{av}^+ remains there, it is not consistent with Stückelberg's proposal mentioned in the beginning. So, we must put $a = i$ for $x_0 < y_0$, in order to take off this contribution. Thus, we obtain correspondence-theoretically

$$(2.4) \quad \Delta^c(x, y) = \Delta_{\text{ret}}^+(x, y) + \Delta_{\text{av}}^-(x, y) = \bar{\Delta}(x, y) + \frac{i}{2} \Delta_1(x, y),$$

if $x_0 \neq y_0$. Here

$$\bar{\Delta}(x) \equiv \frac{1}{2} \varepsilon(x) \Delta(x).$$

When $x_0 = y_0$ (more generally, when the points x and y lie in a space-like relation with each other), we cannot define the Δ^c -function by the correspondence to the classical wave character. This is the essential point for the discussions in Sects. 5 and 7.

Taking into account the identity,

$$\Delta^+(x) = \Delta^-(-x),$$

we get again a relation similar to eq. (2.2)

$$(2.5) \quad \int d^3y u_\beta^*(y) \frac{\overleftrightarrow{\partial}}{\partial y_0} \Delta^-(x, y) = u_\beta^*(x).$$

This equation means that the wave $u_\beta(x)$ with positive frequency at $t = y_0$ comes from the same configuration u_β at $t = x_0 (< y_0)$.

We emphasized above that $\Delta^c(x, y)$ cannot be defined for $x_0 = y_0$ by the correspondence theory. We are interested in what character this undefinedness possesses. On the other hand, in this correspondence-theoretical scheme, the quantum theoretical character is introduced by the fact that the D^c -function has non-zero value also near the light cone, on the contrary to the D_{ret} -function. How can we connect this character to the corpuscular image? For these questions, Fierz's consideration⁽⁶⁾ gives an intuitive image.

Now, let us take the Møller scattering of electrons as an example, following to FIERZ. The matrix element for this process is given by

$$(2.6) \quad \int_{V_x} d^4x \int_{V_y} d^4y \bar{\psi}(x) \gamma_\mu \psi(x) D^c(x - y) \bar{\psi}(y) \gamma_\mu \psi(y).$$

Here, we restrict the emission- and absorption-points x and y in the suitably localized region around some four dimensional space-time points. We express these region by V_x and V_y . $\bar{\psi}(x) \gamma_\mu \psi(x)$ expresses the change of the state of the electron, and the positive (negative) frequency in its Fourier transformation corresponds to the increase (decrease) of the energy of the electron. If we require correspondence-theoretically that *the energy of the quantum carrying action is positive*, such an increase (decrease) of the energy corresponds directly to the absorption (emission) of the quantum.

Let us assume, for definiteness, that V_y spreads around $t = 0$, and, there, the energy of the electron decreases by $h\nu_0$, namely,

$$\bar{\psi}(y) \gamma_\mu \psi(y) = \varrho_\mu(y) \exp \left[i\nu_0 t - \frac{t^2}{T^2} \right].$$

⁽⁶⁾ M. FIERZ: *Helv. Phys. Acta*, **23**, 731 (1950).

Then, we can extend the time-integration in the region V_y to the whole time interval, and we get

$$(2.7) \quad \int_{-\infty}^{\infty} dt \exp \left[i\nu_0 t - \frac{t^2}{T^2} \right] D_{\text{ret}}^{\pm}(x, y) = \begin{cases} 0 & \text{for } D_{\text{ret}}^{-}, \\ \frac{1}{4\pi} \frac{1}{r} \exp \left[i\nu_0(t_x \mp r) - \frac{(t_x \mp r)^2}{T^2} \right], & \text{for } D_{\text{ret}}^{+}. \end{cases}$$

Therefore, the parts which give non-zero contribution to this integral are D_{ret}^{-} and D_{av}^{+} , mathematically. But, we must take the D_{ret}^{+} -function as the D -function in eq. (2.6), physically, if we understand the decrease of the energy at y as the emission of photons. Because we must consider, from the standpoint of corpuscular image, that photons run from past to future.

We can take such a corpuscular image of the photon, only if the region V_x lies in the wave zone. What relation between t_x and T is imposed by this image? The Fourier frequencies of $\exp[-t^2/T^2]$ extend around $\nu = 0$ with the width $2/T$. So, it must be $\nu_0 \gg 1/T$, in order that we can say anything about the sign of the frequency of $\exp[i\nu_0 t - (t^2/T^2)]$. On the other hand, only the part of the region V_x , which lies within the width T around $t_x \approx r$, is affected by the action from the region V_y , as can be seen by eq. (2.7). Thus, we can see that the region V_y is in the wave zone, if $t_x > T$. (Then, $r \gg 1/\nu_0 \sim \lambda_0$, because $r\nu_0 \sim t_x \nu_0 > T\nu_0 \gg 1$). This correspondence-theoretical view has no meaning, if the region V_x is not in the wave zone. This corresponds just to the undefiniteness of $D(x, y)$ for $x_0 = y_0$, in Stückelberg's interpretation.

3. - The causal character of the vacuum- T -function.

In this section, we show that the so-called vacuum- T -function has the causal character in the sense of Stückelberg.

If we assume for the theory to be invariant under the inhomogeneous Lorentz transformation, there exists a displacement operator P_{μ} , and this operator satisfies

$$\hat{\partial} O(x) = i [O(x), P_{\mu}]$$

for the arbitrary operator $O(x)$. We interpret this operator P_{μ} as the energy-momentum operator of the system, and assume that there exists the lowest eigen-value of P_0 . We define the eigenstate corresponding to this value as the vacuum, Φ_0 , and normalize its eigenvalue to be zero. We consider again

scalar field $A(x)$. Then we can write as

$$(\Phi_0, A(x) \Phi_k) = a_{0k} \exp[ikx] \quad k_0 > 0, \quad k_\mu^2 \equiv \mathbf{k}^2 - k_0^2.$$

Now, let us consider the case of the one-particle problem (?). In this case, the vacuum- T -function is given by

$$\langle T(A(x) A(x')) \rangle_0 = i \{ \theta(x_0 - x'_0) \Delta^{(+)'}(x - x') - \theta(x'_0 - x_0) \Delta^{(-)'}(x - x') \};$$

here

$$\Delta^{(+)'}(x - x') = -i \langle A(x) A(x') \rangle_0 = -i \frac{1}{(2\pi)^3} \int \theta(k_0) \varrho(-k^2) \exp[ik(x - x')] d^4k,$$

$$\Delta^{(-)'}(x - x') \equiv i \langle A(x) A(x') \rangle_0 = i \frac{1}{(2\pi)^3} \int \theta(k_0) \varrho(-k^2) \exp[-ik(x - x')] d^4k,$$

$$\theta(a) = \begin{cases} 0 & \text{if } a < 0, \\ 1 & \text{if } a > 0, \end{cases}$$

and

$$\varrho(-k^2) = (2\pi)^3 \sum a_{0k} a_{0k}^*.$$

The $\theta(k^0)$ functions guarantee the positive frequency character of the $\Delta^{(+)'}$ -function, and the negative frequency character of the $\Delta^{(-)'}$ -function. We can see at once that this vacuum- T -function bears Stückelberg's causal character, if we take into account the factor $\theta(x_0)$.

We discuss here a little what is responsible for this causal character, before entering into the many body problem. The essential point to the above consideration is that k_0 is positive definite, and this property comes from the lower-boundedness of the spectrum of the energy operator. But, it is not correct to conclude that the existence of the vacuum warrants the causality. The above analysis means only that we can take the vacuum T function as the function which has the causal character when there exists the vacuum, and this function has not the causal character if P_0 has not the lowest eigenvalue.

If the energy spectrum is not lower-bounded, we can still take some state as the standard state, and make the states with larger eigenvalue than this belong to the positive spectrum, and the states with smaller eigenvalue to the negative spectrum. We can construct a function referring to this standard state, similar to the vacuum- T -function. Here, we can separate $\langle A(x) A(x') \rangle_0$

(?) H. LEHMANN: *Nuovo Cimento*, **11**, 342 (1954).

into two parts

$$\langle A(x)A(x') \rangle_0 = \sum_{n^+} \langle 0 | A(x) | n^+ \rangle \langle n^+ | A(x') | 0 \rangle + \\ + \sum_{n^-} \langle 0 | A(x) | n^- \rangle \langle n^- | A(x') | 0 \rangle.$$

The first part (the positive spectrum part) of the right hand side of this expression gives the causal contribution to the vacuum- T -function, but the contribution from the second part (the negative spectrum part) has anti-causal character. There are two possibilities to make the vacuum- T -function regain the causal character. Firstly, we may introduce such an operator that changes the sign of the energy belonging to the negative spectrum, without changing the sign of the energy in the positive spectrum. Secondly, we may reinterpret the states belonging to the negative spectrum. The positron theory is an example of the second possibility.

We consider an F -function defined in the following, in order to analyse the many body problem ⁽⁸⁾

$$F(x_1, x_2, \dots, x_n) = \langle \Phi_0, A(x_1)A(x_2) \dots A(x_n) \Phi_0 \rangle.$$

The inhomogeneous Lorentz transformation to the scalar field operator $A(x)$ can be written as

$$A(\Lambda x + a) = U(a, \Lambda) A(x) U(a, \Lambda)^{-1}.$$

We suppose that the vacuum state vector Φ_0 remains invariant under this transformation, that is

$$U(a, \Lambda) \Phi_0 = \Phi_0.$$

Then,

$$F(x_1, x_2, \dots, x_n) = F(\Lambda x_1 + a, \Lambda x_2 + a, \dots, \Lambda x_n + a),$$

according to the invariance property under the transformation. Thus the function F is the function of $\xi_i \equiv x_i - x_{i+1}$. We denote the Fourier momentum concerning to this variable ξ_i by p_i ,

$$F(\xi_1, \xi_2, \dots, \xi_{n-1}) = \\ = \int d^4 p_1 d^4 p_2 \dots d^4 p_{n-1} \exp[-i(\xi_1 p_1 + \xi_2 p_2 + \dots + \xi_{n-1} p_{n-1}) \cdot G(p_1, p_2, \dots, p_{n-1})].$$

⁽⁸⁾ A. S. WIGHTMAN: *Phys. Rev.*, **101**, 860 (1956).

In order to make clear the physical meaning of p_i , let us analyse the expression

$$\int \exp[ipa] d^4a F(\xi_1, \dots, \xi_{j-1}, \xi_j + a, \xi_{j+1}, \dots, \xi_{n-1}).$$

If we put $\xi_j = 0$ in this expression, and then rewrite $-a$ as ξ_j , this becomes the Fourier transformation of F with respect to the variable ξ_j . The above expression can be deformed as

$$\begin{aligned} &= \int \exp[ipa] d^4a \langle \Phi_0, A(x_1) \dots A(x_j) A(x_{j+1} - a) \dots A(x_n - a) \Phi_0 \rangle \\ &= \left\langle \Phi_0, A(x_1) \dots A(x_j) \int d^4a \exp[ipa] U(-a, 1) A(x_{j+1}) \dots A(x_n) \Phi_0 \right\rangle. \end{aligned}$$

Thus the above operation takes out the component with the energy-momentum p of the system in the state $\Phi \rangle = A(x_{j+1}) \dots A(x_n) \Phi_0 \rangle$. We can see by this consideration that the Fourier momentum p_i is the possible energy-momentum of the state $A(x_{i+1}) \dots A(x_n) \Phi_0 \rangle$. The energy momentum four-vector is time-like, if we take into account the fact that there is always the center of mass for any physical system. Further, if we assume that the energy spectrum is lower bounded, $G(p_1 \dots p_{n-1})$ has non-zero value only if $p_j^2 \geq 0$ and $p_j^0 > 0$. It must also be noticed in the following consideration that the factor $\exp[-ip(x-y)]$ ($p^2 \geq 0$, $p^0 > 0$) means that the action is propagated by the quantum of positive energy from y to x , as in the case of the one particle problem.

The vacuum T function in the many particle problems,

$$\tau(x_1, x_2, \dots, x_n) = \langle \Phi_0 T(A(x_1) A(x_2) \dots A(x_n)) \Phi_0 \rangle$$

can be expressed by the F -function;

$$\tau(x_1, x_2, \dots, x_n) = F(x_1, x_2, \dots, x_n) = F(\xi_1, \xi_2, \dots, \xi_{n-1})$$

$$\text{for } x_1^0 > x_2^0 > \dots > x_n^0,$$

$$\tau(x_1, x_2, \dots, x_n) = F(x_2, x_1, \dots, x_n) = F(-\xi_1, \xi_2, \dots, \xi_{n-1})$$

$$\text{for } x_2^0 > x_1^0 > \dots > x_n^0.$$

We can see that this τ -function describes the propagation of the action by the quantum of positive energy from x_2 to x_1 , (due to the factor $\exp[-ip\xi_1]$), if $x_1^0 > x_2^0 > \dots > x_n^0$, and from x_1 to x_2 , (due to the factor $\exp[ip\xi_1]$), if $x_2^0 > x_1^0 > \dots > x_n^0$. Thus the vacuum- T -function for the many particle system has causal character in the sense of Stückelberg.

We shall be able to construct the propagation function that has causal character by the same technique as in the case of the one-particle system, if the energy spectrum has not the lowest eigenvalue.

4. - Causal character of the S -matrix.

It is most convenient to analyse the S -matrix by the method of LEHMANN, SYMANZIK and ZIMMERMANN⁽⁹⁾, in order to make clear the causal character of the S -matrix and the assumptions involved in the scheme. Their theory formulates the kinematical part of the S -matrix (or field theory), starting from few and considerably definite assumptions.

Their first assumption is that the field operator reduces to its asymptotic field at $t = \pm \infty$, (asymptotic condition). This asymptotic field may be the clothed wave packet, or may be in the configuration of a bound system (*). This asymptotic condition is formulated explicitly in the case of scalar field, and, there, the bound state is assumed not to appear. The Heisenberg field operator $A(x)$ can be expanded in terms of the complete set of the ortho-normalized positive frequency solution $f_\alpha(x)$ of the Klein-Gordon equation:

$$(4.1) \quad A(x) = \sum_{\alpha} \{ A^{\alpha}(x_0) f_{\alpha}^{*}(x) + A^{\alpha*}(x_0) f_{\alpha}(x) \},$$

where

$$A^{\alpha}(x_0) = i \int_{x_0-i} A(x) \frac{\overleftrightarrow{\partial}}{\partial x_0} f_{\alpha}(x) d^3x.$$

The free (but clothed) fields $A_{in}(x)$ and $A_{out}(x)$ also can be expanded by the same set. The expansion coefficients $A_{in, out}^{*}$, $A_{in, out}$ are time-independent, and can be interpreted as the annihilation-, and creation-operators of incoming (out-

(9) H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **1**, 205 (1955).

(*) When we take into account the bound state, we must construct the operators which create or annihilate the bound systems. It is really possible to construct such operators, and to let their scheme include bound systems. The operator

$$B_n(P) = \int F_n(P, p) d^4p A_{P-p} A_{P+p},$$

can be considered as the creation operator of a bound system composed of the two particles in the state n . Here, P and p are the total and relative four-momentum and F_n is the ortho-normalized eigenfunction of the n -state. This operator satisfies the relation

$$[B_n(P), B_m^{*}(P')] = \delta_{nm} \delta(P - P').$$

going) free particles. The asymptotic condition is expressed as

$$(4.2) \quad \lim_{\tau \rightarrow \mp \infty} (\Phi, A^\alpha(\tau) \Psi) = \Phi, A_{\text{out}}^\alpha \Psi).$$

Here, Φ and Ψ are the arbitrary state vectors in the Heisenberg representation.

This condition corresponds to the integrability condition in the differential formulation of field theory. If we did not impose this condition, their scheme would include such a non-integrable theory as the theory of non-local interaction. But, the non-local interaction is excluded from the theory, as can be seen by the perturbation-theoretical solution of their reduction formula (*). This is closely connected to the situation that the degree of freedom of the system increases if the non-local interaction is switched on. So the asymptotic condition seems not to be satisfied, when we are concerned with the non-local interaction. (See Appendix I.)

They assume, secondly, that the Hilbert space at $t = \pm \infty$ is expanded by the free states. Namely, they assume the existence of the vacuum, which is defined by (+)

$$(4.3) \quad A_{\text{out}}^* \Omega_{\text{out}} = 0$$

and construct the base of Hilbert space by means of A_{out}^α operators, starting from this Ω_{out} .

They further require the condition,

$$(4.4) \quad [A(x), A(y)] = 0 \quad \text{for } (x - y)^2 > 0,$$

as causality. This condition can be replaced by the requirement that the vacuum- T -function is Lorentz covariant, and is not used explicitly.

We can put

$$(4.5) \quad \Omega_{\text{out}} = \Omega_{\text{in}},$$

(*) They assumed that the interaction is of tri-linear form in the first approximation. In this case, we must use the method proposed by KATAYAMA ⁽¹⁰⁾, in order to determine whether the solution is a non-local interaction or a local one. This situation is due to the non-linear character of the interaction. We can see at once that the possible solution is local, in the case where the interaction is bi-linear in the first approximation (see Appendix I). In this case, the possible interactions have the forms, $A(x)^2$ and $A(x) \square A(x)$ (or $\partial_\mu A(x) \cdot \partial_\mu A(x)$), which are local, and the non-local interaction such as $(\square A(x))^2$ is not included in the solution.

(+) This definition is here equivalent to assume the vacuum as the lowest eigenstate of energy-operator.

(10) Y. KATAYAMA: *Prog. Theor. Phys.*, **10**, 31 (1953).

because the vacuum is a collision constant when the interaction between clothed particles is switched on at infinite past and switched off at infinite future adiabatically (*). Then, the S -matrix defined by

$$(4.6) \quad (\Phi_{\text{out}}^{(\alpha)}, \Phi_{\text{in}}^{(\beta)}) = (\Phi_{\text{in}}^{(\alpha)}, S\Phi_{\text{in}}^{(\beta)})$$

can be expressed as

$$(4.7) \quad S = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int \dots \int K_1 \dots K_n \tau(x_1 \dots x_n) : A_{\text{in}}(x_1) \dots A_{\text{in}}(x_n) : d^4x_1 \dots d^4x_n.$$

Here

$$K_i = \square_i - \mu^2,$$

and $:A_{\text{in}}(x_1) \dots A_{\text{in}}(x_n):$ is the so-called normal product of field operators $A_{\text{in}}(x_i)$.

The graphical meaning of the τ -function can be seen, if we transform this function as following,

$$(4.8) \quad \tau(x_1 \dots x_n) = \int \Delta_F(x_1 - \xi_1) \dots \Delta_F(x_n - \xi_n) \sigma(\xi_1, \dots, \xi_n) d^4\xi_1 \dots d^4\xi_n,$$

and substitute this expression into the above S -matrix. Namely, the τ -function gives the matrix element of the Feynman graph (including external lines) for the concerned physical process (+).

Now, this τ -function has a causal character, as was explained in Sect. 3. Thus, the S -matrix also has causal character in the sense of Stückelberg.

The reason why their scheme gives the causal structure to the S -matrix can be divided into two steps. Firstly, the vacuum- T -function appears in the S -matrix. Secondly, the vacuum- T -function has causal character in their model. The essential points for the first step are the asymptotic condition and the fact that Φ_{out} and Φ_{in} belong to the Hilbert space, generated from $\Omega_{\text{in}} = \Omega_{\text{out}}$. At this step, it is not essential that Φ_{in} and Φ_{out} are constructed by means of the configurations with positive frequency (×). The fact that we

(*) Ω_{out} may be different from Ω_{in} by an arbitrary phase factor $e^{i\beta}$. We may put this β as zero, because β is never observed.

(+) This τ -function includes the contributions from the so-called disconnected graphs, and we must subtract such contributions (11). But, this situation is not essential for our discussion.

(×) In fact, if we construct these vectors by means of the configurations of negative frequency, their reduction formulas (accordingly the expression of the S -matrix too) have the same forms as the original ones, provided that we replace f_α 's by f_α^* 's.

(11) R. T. MATTHEWS and A. SALAM: *Proc. Roy. Soc.*, A **221**, 128 (1954); K. NISHIJIMA: *Prog. Theor. Phys.*, **10**, 549 (1953); **12**, 279 (1954); **13**, 305 (1955); Z. MAKI: *Prog. Theor. Phys.*, **15**, 237 (1956).

used only the positive frequency part is concerned to the second step. If there are the configurations of negative frequency, it means that there are the physical states with negative energy, as easily seen from the discussion of the preceding section. In such cases, we must modify the theory, as was expressed in Sect. 3, in order to make the vacuum- T -function regain the causal character. We want to emphasize here that the so-called micro-causality (eq. (4.4)) is not used explicitly, and the covariance requirement to the vacuum- T -function is enough to get the above results.

5. - Causal ambiguity and renormalization constants.

The $\Delta^c(x, y)$ function for $x_0 = y_0$ remains undefined by the correspondence-theoretical standpoint about causality, as emphasized in the previous section. If we use the same function in the case $x_0 = y_0$ as the one defined for $x_0 \neq y_0$, the S -matrix calculated by this function is completely identical with the one which is obtained starting from the differential Schrödinger equation, and includes divergent integrals generally. STÜCKELBERG and his coworker⁽¹²⁾ gave a provisional recipe to treat this divergent integrals. Their method reduces the divergence problem to one of indefiniteness. Further, they showed that such an indefiniteness could be connected to the so-called mass- and charge-renormalizations⁽¹³⁾, and their analysis made way to the concept of the renormalization group⁽¹⁴⁾.

Their recipe can be interpreted clearly by the example, in which a neutral scalar field φ interacts with itself by the φ^3 -interaction, as was shown by them. The S -matrix element for the self-energy graph is given by

$$S^{(2)}(\varphi'' | \varphi') = \frac{1}{2} \int d^4x \int d^4y \varphi''(x) [\Delta^c(x, y)]^2 \varphi'(y).$$

The Fourier expressions for the Δ_1 and $\bar{\Delta}$ -functions are

$$\Delta_1(x) = \frac{1}{(2\pi)^3} \int d^4k \exp[ikx] \delta(k^2 + \mu^2)$$

and

$$\bar{\Delta}(x) = \frac{1}{(2\pi)^4} \int d^4k \exp[ikx] \frac{\mathcal{P}}{k^2 + \mu^2}.$$

⁽¹²⁾ E. C. G. STÜCKELBERG and D. RIVIER: *Helv. Phys. Acta*, **23**, 236 (1949).

⁽¹³⁾ E. C. G. STÜCKELBERG and T. A. GREEN: *Helv. Phys. Acta*, **24**, 153 (1951).

⁽¹⁴⁾ E. C. G. STÜCKELBERG and A. PETERMANN: *Helv. Phys. Acta*, **26**, 499 (1953).

The part of the above matrix-element,

$$\bar{A}A_1 + A_1\bar{A},$$

gives a divergent integral, and the contribution from this part is

$$\Delta^s(k) \sim \int d^4p \left\{ \frac{\delta(p^2 + \mu^2)}{(k-p)^2 + \mu^2} + \frac{\delta((k-p)^2 + \mu^2)}{p^2 + \mu^2} \right\}.$$

Their recipe for this case is as follows. We first re-express this integral in the form in which k 's are included only in the combination of k_μ^2 . Next, we make the power of p in the integrand decrease by differentiating this expression in terms of k_μ^2 , and then carry out the p -integration. At last, we integrate the result with respect to k^2 from zero to k^2 , in order to make the contribution from the convergent part regain the original value. Explicitly calculating, we get in the first step,

$$\sim \int d^4p \int_0^1 du \delta[p^2 + \mu^2 + (u - u^2)k^2],$$

and, by differentiating this with respect to k^2 ,

$$\sim \int_0^1 du (u - u^2) \int d^4p \delta''[p^2 + \mu^2 + (u - u^2)k^2] = \pi \int_0^1 du \frac{u - u^2}{\mu^2 + (u - u^2)k^2}.$$

Their recipe is completed by integrating this expression with respect to k^2 from zero to k^2 , after the u -integration.

In the general case, we must differentiate the concerned matrix element by k^2 n -times, corresponding to the order of the divergence, to get a convergent result of the originally diverging integral. Accordingly, we must integrate the result with respect to k^2 n -times at the last step, to cancel out the effect of the differentiation in the convergent part. By such procedures, there appears an indefinite expression,

$$b_0 + b_1 k^2 + \dots + b_{n-1} k^{2(n-1)},$$

which vanishes by the n -times differentiation by k^2 . If we reform this expression in such a way that k^2 appears in the combination of the wave operator $(k^2 + \mu^2)$, the coefficients of this expansion can be connected to the re-normalization constants of mass and charge.

Now, we translate this recipe into the co-ordinate representation, in order

to see its relation to the discussions in Sect. 2. The differentiation by k^2 corresponds to the multiplication by $(x-y)^2$. So, the differentiation by the necessary times corresponds to make $\Delta^s(x, y)$ finite at $x^0 = y^0$ by the multiplication of $(x-y)^{2n}$, corresponding to the order of the infinity. The k^2 -integration at the last step is carried out in order to make the $(x-y)^2$ -dependence of the convergent part of the concerned integral reduce to the original one. By this procedure, the result coincides with the original value at $x^0 \neq y^0$ (that is, at the region in which the integral has finite contribution). Thus, we see that the above indefiniteness corresponds just to the undefinedness of the $\Delta^s(x, y)$ function at $x^0 = y^0$.

It is interesting to refer to the consideration in the correspondence theory from the point of view of the corpuscular image in Sect. 2. The conditions $t_x > T$ and $\nu_0 \gg 1/T$, which are necessary to make the corpuscular image meaningful, show that T (accordingly t_x , too), needs not to be so large if ν_0 is sufficiently large. This seems to be the natural thing from the standpoint in which we combine the divergence problem to the indefiniteness.

6. - Causality in the dispersion relation.

GOLDBERGER⁽¹⁵⁾ turned his attention to the fact that the scattering amplitude $F_{\alpha\beta}(k; \lambda, \lambda')$ coincided with the non-free part of the Fourier transform $M_{\alpha\beta}(k; \lambda', \lambda)$ of the expression,

$$\lim_{\substack{x_0 \rightarrow +\infty \\ y_0 \rightarrow -\infty}} \langle p\lambda' | [\varphi_\alpha(x), \varphi_\beta(y)] | p\lambda \rangle ,$$

in the domain $k_0 > 0$, apart from a trivial factor. Here, k is the four momentum of the scattered meson, and λ, λ' are variables, other than the momenta p and p' , of the scatter. If we translate the causal character, in the form

$$[\varphi_\alpha(x), \varphi_\beta(y)] = 0 \quad \text{for } (x-y)^2 > 0,$$

into an analytic property of $M_{\alpha\beta}(k; \lambda, \lambda')$ in the complex k -plane, we can use Cauchy's theorem. In this way, we can get a relation—dispersion relation—between the real and imaginary parts of the scattering amplitude.

The above causal property really expresses that the observations at the two space-time points separated spacially from each other do not interfere mutually. But, the property, which is used in the practical analysis, is only that $M_{\alpha\beta}(k; \lambda, \lambda')$ is analytic and bounded in the upper-half of the complex k -plane.

⁽¹⁵⁾ M. L. GOLDBERGER: *Phys. Rev.*, **97**, 508 (1955); **99**, 979 (1955).

We want to emphasize that the causality is used also in another form, in the derivation of the dispersion relation. Namely, the causal character that is discussed in the preceding sections is already included in the structure of $F_{\alpha\beta}(k; \lambda, \lambda')$.

If we neglect this situation, even curious results may appear. These analytic properties may be possessed by rather general models, provided that we do not take into account the fact that the analytic properties come from the commutation relation and use only these properties. For example, even in the theory of non-local interaction (*), in which we cannot construct the scattering amplitude $F_{\alpha\beta}$ consistently, we can derive the dispersion relation, if we give $F_{\alpha\beta}$ suitably. Lee's model is an example of the non-causal model, but the scattering amplitude (¹⁶) has the above analytic property, and we may get the dispersion relation.

The domain of the theory, in which the analytic property necessary to derive the dispersion relation is fulfilled, seems to be wider than the one of causal theory, as we can see from the above discussion. But, the theory that might be constructed by starting from the dispersion relation would not include a wider theory than the present field theory, as far as we require the causality to the theory. Because, the properties of the scattering amplitude, which are imposed by the causal requirement, restrict the possible scheme of the theory rather severely. The utility of the dispersion relation seems only to be a convenient relation which connects the observable quantities or the scattering amplitudes of different physical processes to each other. In fact, the dispersion relation can be derived without referring to the details of the concerned theory, but we must go back to the Lagrangian formalism or some theory equivalent to it which guarantees the consistency between equations of motion, quantization and Schrödinger equation, etc., in order to decide whether the relation is meaningful or not (+).

7. - Concluding discussion.

The structure of field theory can be understood clearly, if we accept the causality in the sense of Stückelberg, as discussed in the precedings sections. Namely, the fundamental assumptions in our field theory are Lorentz covariance, unitarity of the S -matrix, and causality.

(*) In this case, we cannot introduce the M -function in such a way as in the case of causal theory. But, we can define $M_{\alpha\beta}(k; \lambda', \lambda)$ as the analytically continued function of $F_{\alpha\beta}(k; \lambda', \lambda)$ to the complex k_0 value. If this M -function has the concerned analytic property, we can derive the dispersion relation.

(¹⁶) T. OKABAYASHI and S. SATO: *Prog. Theor. Phys.*, **17**, 30 (1957).

(+) Especially, the form of the Born term in the dispersion relation must be brought from the conventional theory.

At this point we want to call attention to the fact that unitarity and causality are independent to each other. All the non-integrable theory are non-causal in the examples, which are proposed up to now ⁽¹⁷⁾. But, we can see this independence from the discussion in the last part of Sect. 5. The perturbation method to construct the S -matrix proposed by STÜCKELBERG ⁽⁵⁾ also explains this situation. They construct the hermitian part of the n -th-order term of the S -matrix by means of the lower order terms, using the unitarity relation, and make up for its antihermitian part in such a way that the propagators included in the hermitian part regain the causal character in their sense.

The interpretation of causality by Stückelberg make clear not only the structure of field theory, but also the conditions which are imposed on the physical system by causality. Further, it throws light on the relation between the limit of the correspondence theory and the difficulties in field theory. We can say nothing whether a further development of field theory is obtainable without modifying the above three assumptions. It might be fruitful to give up the causal requirement at least locally. But, it is very interesting that there remains in the present field theory a part which can be treated only by the mathematical recipe, and which is undetermined physically even by these three assumptions. We succeed temporarily in the treatment of divergent integrals by the renormalization procedure, but we can see by the preceding discussions that this procedure lies out of the frame of the present field theory. On the other hand, this procedure gives satisfactory results at least in quantum electrodynamics. So, we must extend the theory in such a way that the new theory involves the physical contents of this protruded part.

A tentative standpoint in this direction is to improve the fundamental concepts in the quantum field theory, in such a way that we complement the unsatisfactory feature in the causal correspondence, the indefiniteness of the $A'(x, y)$ -function at $x^0 = y^0$. The required element can no longer be classical, but seems to be some quantum theoretical concept.

We construct the present-day quantum field theory on the base of the stable particle image. Our discussions in Sect. 2 and 3 are based on this image. The quantization based on the unstable particle image may be considered instead of the stable one, in order to extend the theory. Then the interaction vertex will behave effectively as a non-local one, though the field operators and their interactions are local. Thus, we may expect that this character makes up the unsatisfactory feature of the physical image at $x^0 = y^0$, to some extent.

⁽¹⁷⁾ For example, E. C. G. STÜCKELBERG and G. WANDERS: *Helv. Phys. Acta*, **27**, 667 (1954); M. E. EBEL: *Dan. Mat. Fys. Medd.*, **29**, No. 2 (1954); W. HEISENBERG: *Zeits. f. Naturf.*, **5a**, 251, 367 (1950); **6a**, 281 (1951); M. FIERZ: *Helv. Phys. Acta*, **23**, 731 (1950).

One may oppose to this intention by the reason that electron and photon concerning our successful quantum electrodynamics are absolutely stable. But, the stable limit in the theory, which is formulated on the base of the unstable particle image, may be different from the one in the conventional theory, with respect to such delicate problems. Further, when we are concerned with the unstable particle, the two formalisms may give considerably different results, quantitatively.

We previously investigated the renormalization problem of unstable particles ⁽¹⁶⁾, in order to get any clue to improve the theory in this direction. We took up this problem on the expectation that the conventional renormalization method would fail, because it was based on the propagation character of stable particles. But, we could find the consistent renormalization method by a little modification ⁽¹⁸⁾, unfortunately or fortunately. We cannot yet find any new concrete clue by studying this problem. The most primitive method might be to expand the field operator by the complete set of the decaying waves, for example $\exp[iEt - \gamma(E)t] \exp[-i\mathbf{k}\mathbf{x}]$, instead of plane waves, and to quantize the expansion coefficients. Such a simple modification would be unsatisfactory and we should have to make any statistical consideration about the ensemble constituted by the unstable particles and their decay products. In such a treatment, we must give a life to the concerned particle from the beginning, and this may be a rather unsatisfactory feature (*).

The other way to improve the theory under the three assumptions mentioned at the beginning of this section will be to utilize the negative energy states suitably. But, the physical interpretation and the consistent formulation of the theory involving these states are very difficult ⁽¹⁹⁾.

At the end of this section, we give the relation between Stückelberg's causality and the so-called micro-causality. The former contains the latter, because the *S*-matrix based on Stückelberg's causality ⁽¹⁾ coincides with the one obtained from the conventional differential form of field theory, apart from the causal ambiguity. It is difficult to combine these two concepts

⁽¹⁸⁾ This problem is treated by a slightly different method by H. ARAKI, Y. MUNAKATA, M. KAWAGUCHI and T. GOTO: *Prog. Theor. Phys.*, **17**, 419 (1957); V. GLASER and G. KÄLLÉN: *Nuclear Physics*, **2**, 706 (1956).

(*) In the conventional theory, the life is given as a function of the coupling constant of the concerned interaction. To the contrary, the coupling constants may be given as a secondary quantity determined by the life. The decay constant γ can be interpreted as the imaginary part of the complex mass. Then, some of the coupling constants are absorbed in the complex mass. If we can construct the theory in such a way, it is not unsatisfactory to give the life at the beginning.

⁽¹⁹⁾ P. A. M. DIRAC: *Proc. Roy. Soc.*, **180**, 1 (1942); W. PAULI: *Rev. Mod. Phys.*, **15**, 175 (1943); W. HEISENBERG: *Nachr. Göttingen Nat. Wiss.*, Nr. 8 (1953); *Zeits. f. Naturf.*, **9a**, 272 (1954); **10a**, 425 (1955); **12a**, 177 (1957); *Zeits. f. Phys.*, **144**, 1 (1956); *Nuclear Phys.*, **4**, 532 (1957).

directly, because Stückelberg's causality is expressed in terms of the field quantities at the different times, and the micro-causality is formulated at one instant. We can give the correspondence only in the formal limiting process, as following:

$$\begin{aligned}
 \langle 0 | [A(x), A(x')] | 0 \rangle_{x_0 = x'_0} &= \langle 0 | \left\{ \lim_{x_0 \rightarrow x'_0 + \varepsilon} T(A(x)A(x')) - \lim_{x_0 \rightarrow x'_0 - \varepsilon} T(A(x)A(x')) \right\} | 0 \rangle \\
 &= \lim_{x_0 \rightarrow x'_0 + \varepsilon} \Delta^c(x, x') - \lim_{x_0 \rightarrow x'_0 - \varepsilon} \Delta^c(x, x') \\
 &= \lim_{t \rightarrow +0} \Delta_{\text{ret}}^+(\mathbf{r}, t) - \lim_{t \rightarrow -0} \Delta_{\text{av}}^-(\mathbf{r}, t) \\
 &= \lim_{t \rightarrow +0} \Delta_{\text{ret}}^+(\mathbf{r}, t) - \lim_{t \rightarrow -0} \Delta_{\text{ret}}^+(\mathbf{r}, -t) \\
 &= 0.
 \end{aligned}$$

APPENDIX I

Non-local interaction (*).

The basic defect of non-local interactions is that we have no relativistic method to formulate them consistently. If we use the relativistically invariant non-local interactions, the theory becomes non-integrable. Because such interactions involve the form functions extending also in the time-like direction. In this appendix we start from the investigation about the system described by the Lagrangian which contains the higher order derivatives of the dynamical variable, in order to understand this situation as intuitively as possible.

The non-local interaction can be considered generally as such a limiting case that the order of the derivatives of field variables contained in the interaction becomes infinitely large. For instance, the form function F in the interaction between a spinor field $\psi(x)$ and a pseudoscalar field $\varphi(x)$,

$$ig \int \bar{\psi}(x') F(x', x'', x''') \varphi(x'') \gamma_5 \psi(x''') dx' dx'' dx''',$$

can be expressed as

$$\begin{aligned}
 F(x', x'', x''') &= \frac{1}{(2\pi)^8} \int G(L, l) \exp \left[iL \left(\frac{x' + x'''}{2} - x'' \right) \right] \exp [il(x' - x''')] dL dl = \\
 &= G \left(i \frac{\partial}{\partial x''}, i \frac{\partial}{\partial x'''} \right) \delta(x' - x'') \delta(x' - x''').
 \end{aligned}$$

(*) The contents of this section were a part of the article published in the *Soryū-shiron-kenkyū* (mimeographed circular in Japanese), **5**, 49 (1953).

Here, the invariance of $F(x', x'', x''')$ under the inhomogeneous Lorentz transformations is taken into account. There are different properties between the case, where we are concerned with the derivatives of finite order, and the case of the derivatives of infinite order, as will be seen later. But, these differences seem to have nothing to do with the discussion about the integrability.

For simplicity, we begin the discussion with the particle dynamics in which the dynamical variable is q . The transformation function from the time t_1 to t_N is given by

$$K(N, 1) = \int \dots \int \exp[iS] d_{(\text{path})},$$

here

$$S = \int L(\dot{q}, q) dt, \quad \text{and} \quad \dot{q} = \frac{dq(t)}{dt},$$

when the Lagrangian of the system is described by $L(\dot{q}, q)$ ⁽²⁰⁾. If we divide the time interval $t_N - t_1$ into small divisions whose intervals are ε , and express the velocity at each divided time as $\dot{q}_i = \lim_{\varepsilon \rightarrow 0} (q_{i+1} - q_i)/\varepsilon$, we can reform the integral in S into the sum,

$$K(N, 1) = \int \dots \int \exp \left[i \sum_{i=1}^{N-1} L(q_{i+1}, q_i) \right] \frac{dq_2}{A} \frac{dq_3}{A} \dots \frac{dq_{N-1}}{A},$$

where A is a normalization constant. We can split the sum into two parts in such a way that the one contains only variables q_i ($i \geq k$) and the other contains q_j ($j \leq k$), that is, into $\sum_{i=k}^{N-1} L(q_{i+1}, q_i)$ and $\sum_{j=1}^{k-1} L(q_{j+1}, q_j)$. By this splitting we have the composition law,

$$K(N, 1) = \int K(N, k) K(k, 1) \frac{dq_k}{A}.$$

Namely, we can solve the problem, by giving an initial value q_1 at the time t_1 , in the two steps, first from t_1 to t_k and then from t_k to t_N . Here we can take arbitrary times as t_k , and this situation corresponds to the integrability in field theory.

When the Lagrangian contains \ddot{q} , we must treat $L(q_{i+1}, q_i, q_{i-1})$ instead of $L(q_{i+1}, q_i)$, contrarily to the ordinary dynamics (because $\ddot{q}_i = \lim_{\varepsilon \rightarrow 0} (\dot{q}_{i+1} - \dot{q}_i)/\varepsilon$).

Then the expression S becomes $\sum_{i=k}^{N-1} L(q_{i+1}, q_i, q_{i-1}) + \sum_{j=1}^{k-1} L(q_{j+1}, q_j, q_{j-1})$, by the same procedure as used above. The first term contains q_i ($i \geq k-1$), and the second term contains q_j ($j \leq k$). Thus, the composition law fails if we take only q as the dynamical variable, on account of the situation that the time

⁽²⁰⁾ R. P. FEYNMAN: *Rev. Mod. Phys.*, **20**, 367 (1948).

division $t_{k-1} < t < t_k$ concerns with both terms. We must treat not only q as the dynamical variable, but also \dot{q} , in order to get the composition law again (*).

Next, we consider the system, whose Lagrangian is composed of the two parts,

$$L = L_0(\dot{q}, q) + L_i(\ddot{q}, \dot{q}, q).$$

q is the complete dynamical variable for the part L_0 , but we must take into account \dot{q} also as dynamical variable to treat the whole Lagrangian. Thus, we cannot consider the term L_i as a perturbation.

The higher the order of the derivatives involved in the Lagrangian, the broader the overlapping between $K(N, k)$ and $K(k, 1)$. This overlapping becomes the whole time interval in the limit of infinite order. This case corresponds to the non-local interaction, with respect to this situation.

The case, where the highest order of the concerned derivatives is finite, is different from the case of the infinite order derivative, in another respect. For definiteness, we take the system described by the Lagrangian

$$L = q F \left(\frac{d}{dt} \right) q,$$

where F is supposed to be

$$F \left(\frac{d}{dt} \right) = \exp \left[f \left(\frac{d}{dt} \right) \right] \prod_i^N \left(1 + \frac{1}{\omega_i^2} \frac{d^2}{dt^2} \right).$$

The second factor can be rewritten as an ensemble of N harmonic oscillators⁽²²⁾, and the first factor cannot be connected to the harmonic oscillator model. If the highest order of the derivatives is finite, we can always factorize F as in the second factor. But, there appears such a factor that we cannot reduce to the harmonic oscillator as the first factor, when we concern with the Lagrangian involving derivatives of infinite order. In field theory, we quantize the Fourier coefficient of field quantity, on the base of the corpuscular image, when the coefficient constitutes a harmonic oscillator. Thus, the part, corresponding to the first factor in the above example, does not increase the degree of freedom of the field in this sense. But, the situation concerning with the composition law is not different from the preceding discussion. This can be seen also from the analysis of Pais-Uhlenbeck.

PAULI⁽⁴⁾ classified form factors into two classes, normal class and abnormal class. If the manifold of the solutions of the system with the non-

(*) We have the method of Ostrogradsky⁽²¹⁾ to treat both q and \dot{q} as dynamical variables. This method is modified by Pais and Uhlenbeck⁽²²⁾ so as to adapt it to the harmonic oscillator model. If we extend Ostrogradsky's method to field theory, there appears a new difficulty in the case of non-local interactions, as will be shown in Appendix II.

⁽²¹⁾ E. T. WHITTAKER: *Analytical Dynamics* (London, 1937), fourth edition, p. 265.

⁽²²⁾ A. PAIS and G. E. UHLENBECK: *Phys. Rev.*, **79**, 145 (1950).

local interaction is the same as the one of the free system, the form factor contained in the interaction is defined to belong to the normal class. The form factor which does not have such a property belongs to the abnormal class. According to this definition, the interaction with the form factor belonging to the normal class does not increase the degree of freedom of the field, and has such a property as the first factor in the above example. In soluble simple problems, we can find the form function of the normal class, actually. For instance, in the equation of motion

$$\ddot{q}(t) + q(t) = g \int \Phi(t-t') q(t') dt',$$

such form functions as

$$\Phi(t) = -\frac{1}{a^2 + t^2} \quad (a > 0)$$

and

$$\Phi(t) = -\frac{1}{a} \sqrt{\frac{\pi}{2}} \exp[-a|t|] \quad (a > 0)$$

belong to the normal class. But, the problem of the integrability of the theory is not solved only by the existence of form factors of this class.

In the case of non-local interaction, we cannot pick up higher derivatives as dynamical variables, because higher derivatives do not appear explicitly. On the other hand, we cannot specify the state at the sharply defined time in this case. Thus we must pursue the S -matrix connecting the asymptotic form in the infinite past to the one in the infinite future, without following the sharp times step by step (*). The form functions of the abnormal class must be excluded also in this method, because the S -matrix ought to be unitary. It is not proved that we cannot construct the unitary S -matrix in the case of the form function belonging to the normal class. But we cannot find such a method yet, and it seems to be impossible, as will be anticipated from the preceding discussions. We cannot determine whether the concerned form factor belongs to the normal class or to the abnormal one, until the solutions are obtained explicitly. So, it seems to be hopeless to get such an inclusive formulation that this difference is implied from the beginning, in place of Lagrangian formalism (+).

Here we explained the situation about the integrability by the simple particle dynamics. The situation in field theory is clearly seen by these discussions, already. The scalar field, interacting with itself through bilinear inter-

(*) In the theory of non-local interactions, the conservation law can be obtained only in the integrated form. On the other hand, in the theory involving higher derivatives of field, we can derive the conservation law of differential form. (See Appendix II).

(+) In the local theory, the consistency between wave equations, commutation relations, Schrödinger equation and conservation laws etc., is automatically guaranteed in the Lagrangian formalism (2). It will be necessary to construct any formulation that plays such a role.

action, corresponds to the particle dynamics, as it is. For the case of spinor field, and of multi-linear (except for bilinear) interaction, we must investigate in detail whether the concerned interaction is local or non-local⁽¹⁰⁾. For instance, we can see at once from our discussion that the interaction of the form, $(\square A(x))^2$, belongs to the interaction with higher derivative, as already mentioned in the footnote of Sect. 4. We must bear in mind that the form functions of normal class also are excluded in the scheme of Lehmann-Symanzik-Zimmermann⁽⁹⁾ discussed in Sect. 4. It is due to the fact that we imposed the causality on the vacuum T -functions implicitly as will be seen from the discussions in the text. The form functions of abnormal class are excluded owing to the asymptotic condition.

It seems to be impossible to construct any satisfactory formulation for non-local interactions. If experiments⁽²³⁾ suggest some non-local interaction, we may interpret this character as a secondary effect through some intermediate field. (Coulomb force and nuclear force have such character). If we aim at avoiding divergences by non-local interactions, we may consider such a possibility as following. The order of divergence in the self-energy of electron was diminished by the appearance of the new degrees of freedom, σ - and ϱ -matrix, and by using the hole theory. Now, we have new degrees of freedom concerning the iso-space, which are introduced in the phenomena of the so-called strange particles. If these degrees of freedom are connected with the Minkowski space in any sense, the situation concerning divergences will be altered^(*) ⁽²⁴⁾. The other possibility might be to construct a field theory on the modified particle image, as mentioned in Sect. 7.

⁽²³⁾ For example, R. HOFSTADTER: *Rev. Mod. Phys.*, **28**, 214 (1956); D. R. YENNIE, M. M. LEVY and D. G. RAVENHALL: *Rev. Mod. Phys.*, **29**, 144 (1957).

^(*) In this possibility, it is difficult to introduce the so-called universal length⁽²⁵⁾. The aim to introduce the universal length was not merely to formulate cut-off (or to take into account the classical extension of particles) in field theory, but it was expected for new duality to appear at the small dimension. Non-local interaction also is only a temporal technique, in this respect.

⁽²⁴⁾ T. OKABAYASHI: *Prog. Theor. Phys.*, **10**, 499 (1953). The situations about the strange particles were not clear at that time. Recently, have appeared new experimental facts that seem to support the existence of the correlation between the Minkowski space and iso-space. In the strong interaction, the hyperonic charge, the isotopic spin, and the parity are conserved. In the weak interaction, the first two quantities are not conserved, and the parity seems not to be conserved in almost all weak interactions.

C. S. WU, E. AMBLER, R. W. HAYWARD, D. D. HOPPEs, and R. P. HUDSON: *Phys. Rev.*, **105**, 1413 (1957); R. L. GARWIN, L. M. LEDERMAN and M. WEINRICH: *Phys. Rev.*, **105**, 1415 (1957); F. EISLER, R. PLANO, A. PRODELL, N. SAMIOS, M. SCHWARTZ, J. STEINBERGER, P. BASSI, V. BORELLI, G. PUPPI, H. TANAKA, P. WALOSCHEK, V. ZOBOLI, M. CONVERSI, P. FRANZINI, I. MANNELLI, R. SANTANGELO, V. SILVESTRINI, D. A. GLASER, C. GRAVES and M. L. PERI: *Phys. Rev.*, **108**, 1353 (1958).

An analysis in which these situations are taken into account, is reported in our recent article published in *Prog. Theor. Phys.* **20**, (1958).

⁽²⁵⁾ W. HEISENBERG: *Ann. der Phys.*, **32**, 533 (1936).

APPENDIX II

A Note on the Lagrangian including higher derivatives⁽⁺⁾.

We must take account not only of q , but also of its higher derivatives as dynamical variables in order to keep the composition law, when the Lagrangian of the system involves the higher derivatives of q . OSTROGRADSKY proposed a method to this problem in particle dynamics. In this appendix, we extend his method to field theory. Then, there appears a new contribution to the angular momentum, as will be expected. We restrict ourselves to the variation principle in classical field theory. The transition to the quantum theory is straightforward. Further, we suppose that there is only one scalar field $\varphi(x)$, for the extension to more general cases, where many fields with components interact with each other, is trivial.

We consider the system described by the Lagrangian,

$$\mathcal{L} = \mathcal{L} \left(\varphi, \frac{\partial \varphi}{\partial x^\mu}, \frac{\partial^2 \varphi}{\partial x^\mu \partial x^\nu}, \dots, \frac{\partial^n \varphi}{\partial x^\mu \partial x^\nu \dots} \right).$$

We denote the virtual variation of φ by $\delta^* \varphi$, and the variation of various quantities due to this variation by adding δ^* in front of them. The notation δ means the variation that comes from the infinitesimal transformation of space-time co-ordinates and field quantity. The virtual variation is commutative with the operation $\partial/\partial x^\mu$,

$$\delta^* \frac{\partial \varphi}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \delta^* \varphi,$$

on the contrary to δ . The total variation of φ consists of the virtual variation $\delta^* \varphi$ and the variation, which comes from the infinitesimal transformation of the co-ordinates,

$$(A.II.1) \quad \delta \varphi = \delta^* \varphi(x) + \frac{\partial \varphi}{\partial x^\mu} \delta x^\mu.$$

The field equation can be obtained from the stationary property of the action integral to the virtual variation, which vanishes at the boundary \bar{X} of the concerned four dimensional domain X .

$$\delta^* I = \delta^* \int_X \mathcal{L} dX = \int_X \delta^* \mathcal{L} dX = 0.$$

(+) The contents of this appendix were published in the *Soryūshiron-kenkyū* (mimeographed circular in Japanese), 4, No. 9, 37 (1952).

Here $\delta^*\mathcal{L}$ can be proved, by the induction method, to be

$$(A.II.2) \quad \delta^*\mathcal{L} = [\mathcal{L}]_\varphi \delta^*\varphi + \frac{\partial}{\partial x^\mu} \mathcal{M}^\mu,$$

where

$$(A.II.3) \quad [\mathcal{L}]_\varphi \equiv \sum_{m=0}^n (-1)^m \frac{\partial^m}{\partial x^\mu \partial x^\nu \dots} \frac{\partial \mathcal{L}}{\partial (\partial^m \varphi / \partial x^\mu \partial x^\nu \dots)},$$

$$(A.II.4) \quad \mathcal{M}^\mu \equiv \sum_{t=0}^{n-1} \pi^{\mu\nu\dots} \delta^* \frac{\partial^t \varphi}{\partial x^\nu \dots}$$

$$(A.II.5) \quad \pi^{\mu\nu\dots} \equiv \sum_{s=0}^{n-t-1} c(s+t+1, t) \frac{\partial^s}{\partial x^\sigma \dots} \frac{\partial \mathcal{L}}{\partial (\partial^{s+t+1} \varphi / \partial x^\mu \partial x^\nu \dots \partial x^\sigma \dots)},$$

and

$$(A.II.6) \quad c(m, t) = \sum_{r=0}^{m-t-1} (-1)^r \binom{m}{r} \binom{m-r-1}{t}.$$

If we make the second term of $\delta^*\mathcal{L}$ vanish by the partial integration, we get the equation of motion,

$$(A.II.7) \quad [\mathcal{L}]_\varphi = 0.$$

In order to obtain conservation laws, we impose that the action integral is stationary under the infinitesimal variation of x and φ which includes the variation of the boundary,

$$\delta I = \int_{\bar{x}} \delta^* \mathcal{L} dX + \int_{\bar{x}} \mathcal{L} \delta x^\mu d\bar{X}_\mu = \int_{\bar{x}} \left\{ \delta^* \mathcal{L} + \frac{\partial}{\partial x^\mu} (\mathcal{L} \delta x^\mu) \right\} dx = 0.$$

Then, we get

$$\delta^* \mathcal{L} + \frac{\partial}{\partial x^\mu} (\mathcal{L} \delta x^\mu) = 0.$$

This equation can be rewritten in the form

$$(A.II.8) \quad [\mathcal{L}]_\varphi \delta^* \varphi = \frac{\partial}{\partial x^\mu} \mathcal{U}^\mu,$$

where

$$(A.II.9) \quad \mathcal{U}^\mu = -\mathcal{M}^\mu - \mathcal{L} \delta x^\mu = -\sum_{t=0}^{n-1} \pi^{\mu\nu\dots} \delta \frac{\partial^t \varphi}{\partial x^\nu \dots} + \mathcal{C}_\sigma^\mu \delta x^\sigma,$$

and

$$(A.II.10) \quad \mathfrak{T}_\sigma^\mu \equiv \sum_{t=0}^{n-1} \pi^{\mu\nu\dots} \frac{\partial}{\partial x^\sigma} \frac{\partial^t \varphi}{\partial x^\nu \dots} - \mathcal{L} \delta_\sigma^\mu.$$

Here, eq. (A.II.1), (A.II.2), (A.II.4) are used. The right hand side of eq. (A.II.8) vanishes, owing to the equation of motion (A.II.7),

$$(A.II.11) \quad \frac{\partial \mathcal{U}^\mu}{\partial x^\mu} = 0.$$

This equation expresses the conservation of \mathcal{U}^μ .

The conservation law of energy-momentum is obtained by the infinitesimal translation of co-ordinates,

$$\begin{aligned} \delta x^\mu &= \varepsilon^\mu = \text{const}, \\ \delta \left(\frac{\partial^t \varphi}{\partial x^\sigma \dots} \right) &= 0, \end{aligned} \quad 0 \leq t \leq n-1.$$

In this case, we have

$$\mathcal{U}^\mu = \mathfrak{T}^\mu_\nu \delta x^\nu,$$

and (A.II.11) becomes

$$\frac{\partial \mathfrak{T}^\mu_\nu}{\partial x^\mu} = 0.$$

\mathfrak{T}^μ_ν is the energy-momentum density.

The infinitesimal homogeneous Lorentz transformation gives rise to the conservation of angular momentum. The concerned variation is

$$\begin{aligned} \delta x^\mu &= \varepsilon^\mu_\nu x^\nu, & (\varepsilon_{\mu\nu} &= -\varepsilon_{\nu\mu}) \\ \delta \varphi &= 0, \\ \delta \left(\frac{\partial^t \varphi}{\partial x^\mu \partial x^\nu \dots} \right) &= \varepsilon_\mu^\sigma \varepsilon_\nu^\rho \frac{\partial^t \varphi}{\partial x^\sigma \partial x^\rho \dots} \end{aligned} \quad (t \neq 0).$$

Then \mathcal{U}^μ is given by

$$\begin{aligned} \mathcal{U}^\mu &= \varepsilon^\nu_\sigma (x^\sigma \mathfrak{T}^\mu_\nu - \mathcal{P}^{\mu\sigma}_\nu) \\ &= \frac{1}{2} \varepsilon^{\nu\sigma} (\mathfrak{T}^\mu_{[\nu} x_{\sigma]} + \mathcal{P}^\mu_{[\nu\sigma]}) \\ &\equiv \frac{1}{2} \varepsilon^{\nu\sigma} \mathcal{M}^\mu_{\nu\sigma}, \end{aligned}$$

where (*)

$$\mathcal{P}^{\mu\nu}_\sigma = \sum_{t=1}^{n-1} \pi^{\mu\nu\rho\dots} \varepsilon^\lambda_\rho \varepsilon \dots \frac{\partial^t \varphi}{\partial x^\sigma \partial x^\lambda \dots}.$$

(*) $A_{[\mu,\nu]} \equiv A_{\mu\nu} - A_{\nu\mu}.$

The equation (A.II.8) becomes

$$\frac{\partial \mathcal{M}^{\sigma}_{\mu\nu}}{\partial x^{\sigma}} = [\mathcal{L}]_{\varphi} \frac{\partial \varphi}{\partial x^{[\mu}} \cdot x_{\nu]} = 0.$$

$\mathcal{M}^{\sigma}_{\mu\nu}$ is the angular momentum density, and its first term is the orbital angular momentum density. If the Lagrangian does not involve the higher derivatives of φ , there is not the second term of $\mathcal{M}^{\sigma}_{\mu\nu}$. The reason why this term appears will be seen intuitively as follows. In the case that the Lagrangian contains the higher derivatives of φ , (here the highest order is $2n$), we need also higher derivatives of φ , in order to specify the motion. In fact, the equation (A.II.9) shows that we need $\partial^r \varphi / \partial x^{\mu} \dots$ ($0 \leq r \leq n-1$) ⁽⁺⁾, in order to express the action integral by means of the quantities on the initial and final space-like surfaces, as will be expected from the analogy with the conventional theory ⁽²⁾. Now, these quantities are tensors of higher rank, though φ is scalar. The second term of $\mathcal{M}^{\sigma}_{\mu\nu}$ shows that these tensors give contributions to spin angular momentum density independently, corresponding to each rank. Such a situation will give raise to some difficulty in the theory of non-local interaction, because the clothed particle will have the spin different from the one of the bare particle.

We did not take into account the corpuscular image based on the harmonic oscillator model, in the above method. If we want to stand on this model we must use the method of Pais-Uhlenbeck ⁽²²⁾.

(⁺) Strictly speaking, these are to be $\partial^r \varphi / \partial t^r$, for we cannot give the value of φ and $\partial \varphi / \partial x^i$ ($i = 1, 2, 3$) independently.

Multiple Scattering by a Random Stack of Dielectric Slabs (*).

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CONTENTS. — 1. Introduction. — 2. Formulation of the problem. — 3. The stochastic majorants. — 4. The binomial moments. 1. The Gaussian estimate of $\langle \hat{\beta}_N^{2n} \rangle$. 2. The exponential estimate of $\langle \hat{\beta}_N^{2n} \rangle$. — 5. Analysis of multiple scattering. 1. Mean square reflection coefficient; $AN=0.99$, $N=20$. 2. Mean square transmission coefficient; $AN=0.50$, $N=250$, $A^2N=10^{-3}$. — 6. Conclusion.

1. — Introduction.

In treating problems involving the scattering of incident radiation (*e.g.*, sound waves, radio waves) by a random medium, it has long been recognized that the random nature of the scattering medium plays an important part in attenuating multiple scattering. A familiar qualitative explanation of this phenomenon is that the internal field produced in the medium by single scattering is considerably weakened by the «incoherence» or, to put it differently, that the contributions to the internal field from uncorrelated parts of the medium add like power rather than like amplitude. However, with the exception of the work of MINTZER ⁽¹⁾, there appears to be little in the way of explicit calculations of the strength of multiple scattering in a random medium. The difficulty seems to stem mainly from the three-dimensional nature of the usual

(*) The research reported in this paper has been sponsored by the Electronics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, under Contract no. AF 19(604)3495.

⁽¹⁾ D. MINTZER: *Journ. Acoust. Soc. Am.*, **25**, 1107 (1953).

scattering problems, which requires one to carry out difficult multiple integrations in order to estimate the high order scattering.

In the present paper, we treat the multiple scattering of incident plane waves by a random stack of dielectric slabs. Because of the one-dimensional nature of the problem, and because of the simple way in which the random stack is generated, we are able to carry the analysis of the effects of multiple scattering quite far. In particular, we find that we can get an upper bound for the strength of scattering of any order which, unlike the crude estimates usually given, takes into account the random nature of the scattering medium. Our basic result is that when the number of slabs is large, the Neumann series for the mean square transmission and reflection coefficients converge much more rapidly than would be inferred from the smallness of the perturbation of the incident field by the scattering medium. This additional attenuation of multiple scattering is directly attributable to the random nature of the scattering medium. However, it is found that the randomness of the scattering medium is no longer effective in reducing multiple scattering when the order of the scattering exceeds the number of slabs in the stack.

Despite the evident difficulty of extending the analysis to three-dimensional scattering problems and more complicated scattering media, there is every reason to believe that our qualitative conclusions are quite generally valid.

2. - Formulation of the problem.

Consider a stack of N dielectric slabs, all of the same thickness d . Choose the origin of a Cartesian system in the lower face of the bottom slab, and let the z -axis point upwards in the direction normal to the slabs (*). The configuration is illustrated in Fig. 1.

The random function $\varepsilon(z)$ which describes the dielectric constant profile of the stack of N slabs is constructed as follows. Let β be a Bernoulli random variable, which assumes the values -1 and $+1$ with equal probability, *i.e.*, with probability $\frac{1}{2}$. To obtain a realization (sample function) of the random function $\varepsilon(z)$ we perform a sequence of N independent trials, each consisting of an observation of β , and we denote the outcomes of these trials by $\beta(1, \alpha)$, $\beta(2, \alpha)$, ..., $\beta(N, \alpha)$. (The index α ranges over a sample space S consisting of 2^N points, each of which

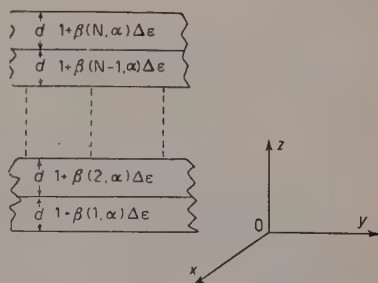


Fig. 1. - The scattering medium (schematic).

(*) The slabs are regarded as infinite in the x and y directions.

corresponds to a possible outcome of the sequence of N trials). Then the realization indexed by α of the random function $\varepsilon(z)$ is given by the piecewise constant function (*)

$$(1) \quad \varepsilon(z, \alpha) = 1 + \Delta\varepsilon(z, \alpha) = \begin{cases} 1 & , & z < 0, \\ 1 + \beta(r, \alpha)\Delta\varepsilon, & (r-1)d \leq z < rd, 1 \leq r \leq N, \Delta\varepsilon > 0, \\ 1 & , & z \geq Nd. \end{cases}$$

For example, one might toss a well-balanced coin and assign one of the slabs the dielectric constant $1 + \Delta\varepsilon$ if the outcome is a head, and $1 - \Delta\varepsilon$ if the outcome is a tail, then toss the coin again and assign any other slab the dielectric constant $1 + \Delta\varepsilon$ if the outcome is a head and $1 - \Delta\varepsilon$ if the outcome is a tail, and so on until all N slabs have been assigned values of the dielectric constant. We shall often indulge in the usual abuse of notation, whereby we suppress the index α and use the same symbol to denote both a random variable and the values it assumes, or to denote a random function and its realizations.

Suppose now that a scalar plane wave $\exp[ikz]$ is incident from below on a random stack of slabs generated in the manner just described, and that the resulting field $u(z)$ obeys the inhomogeneous wave equation (+)

$$(2) \quad \left[\frac{d^2}{dz^2} + k^2 \right] u(z) = -k^2 \Delta\varepsilon(z) u(z).$$

Since $\Delta\varepsilon(z)$ is a random function, the same is true of $u(z)$; the index α has been suppressed, so that (2) can be regarded either as a stochastic differential equation, or as the wave equation obeyed separately by each realization of $u(z)$. The integral equation corresponding to (2) is

$$(3) \quad u(z) = \exp[ikz] - k^2 \int_0^{Nd} G(z, z') \Delta\varepsilon(z') u(z') dz',$$

where

$$G(z, z') = \frac{1}{2ik} \exp[ik|z - z'|],$$

is the one-dimensional free-space Green's function, and the incident wave $\exp[ikz]$ is, of course, a solution of the homogeneous wave equation. In writing the limits of integration, we have used the fact that by construction the random

(*) To avoid confusion, note that Δ is used exclusively to designate an increment of the quantity it precedes, and is never used as an independent symbol.

(+) It is also assumed that $u(z)$ and $du(z)/dz$ are continuous. For simplicity we have chosen the dielectric constant of the medium in which the stack is immersed to be unity, with the result that $\varepsilon(z, \alpha)$ can be less than unity. Those who are disturbed by this can replace $\varepsilon(z, \alpha)$ in (1) by $\varepsilon(z, \alpha)/\varepsilon_0$, where $\varepsilon_0 > 1$ and $\varepsilon_0(1 - \Delta\varepsilon) \geq 1$; equation (2) remains the same.

function $\Delta\epsilon(z)$, which is the dielectric constant profile *centered* about its mean, vanishes outside of the interval $0 \leq z < Nd$. The formal solution of (3) is given by the Neumann series

$$(4) \quad \left\{ \begin{aligned} u(z) &= \exp[ikz] + \frac{ik}{2} \int_0^{Nd} \exp[ik|z-z_1|] \exp[ikz_1] \Delta\epsilon(z_1) dz_1 - \\ &- \frac{k^2}{4} \int_0^{Nd} \int_0^{Nd} \exp[ik|z-z_1|] \exp[ik|z_1-z_2|] \exp[ikz_2] \Delta\epsilon(z_1) \Delta\epsilon(z_2) dz_1 dz_2 + \dots \\ &= 1 + \sum_{n=1}^{\infty} \left(\frac{ik}{2} \right)^n \int_0^{Nd} \int_0^{Nd} \dots \int_0^{Nd} \exp[ik|z-z_1|] \exp[ik|z_1-z_2|] \dots \exp[ik|z_{n-1}-z_n|] \cdot \\ &\quad \cdot \exp[ikz_n] \Delta\epsilon(z_1) \Delta\epsilon(z_2) \dots \Delta\epsilon(z_n) dz_1 dz_2 \dots dz_n, \end{aligned} \right.$$

where, for the moment, we leave aside the question of convergence.

It is particularly convenient in one-dimensional problems of this kind to describe the solution $u(z)$ in terms of a transmission coefficient T and a reflection coefficient R , defined by the relations

$$(5) \quad \left\{ \begin{aligned} u(z) &= T \exp[ikz], & z \geq Nd, \\ u(z) &= \exp[ikz] + R \exp[-ikz], & z < 0. \end{aligned} \right.$$

It follows from (4) and (5) that T and R are given by

$$(6) \quad \left\{ \begin{aligned} T &= 1 + \frac{ik}{2} \int_0^{Nd} \Delta\epsilon(z_1) dz_1 - \frac{k^2}{4} \int_0^{Nd} \int_0^{Nd} \exp[-ikz_1] \exp[ik|z_1-z_2|] \exp[ikz_2] \cdot \\ &\quad \cdot \Delta\epsilon(z_1) \Delta\epsilon(z_2) dz_1 dz_2 + \dots \\ &= 1 + \sum_{n=1}^{\infty} \left(\frac{ik}{2} \right)^n \int_0^{Nd} \int_0^{Nd} \dots \int_0^{Nd} \exp[-ikz_1] \exp[ik|z_1-z_2|] \dots \exp[ik|z_{n-1}-z_n|] \cdot \\ &\quad \cdot \exp[ikz_n] \Delta\epsilon(z_1) \Delta\epsilon(z_2) \dots \Delta\epsilon(z_n) dz_1 dz_2 \dots dz_n, \end{aligned} \right.$$

and

$$(7) \quad \left\{ \begin{aligned} R &= \frac{ik}{2} \int_0^{Nd} \exp[2ikz_1] \Delta\epsilon(z_1) dz_1 - \frac{k^2}{4} \int_0^{Nd} \int_0^{Nd} \exp[ikz_1] \exp[ik|z_1-z_2|] \exp[ikz_2] \cdot \\ &\quad \cdot \Delta\epsilon(z_1) \Delta\epsilon(z_2) dz_1 dz_2 + \dots \\ &= \sum_{n=1}^{\infty} \left(\frac{ik}{2} \right)^n \int_0^{Nd} \int_0^{Nd} \dots \int_0^{Nd} \exp[ikz_1] \exp[ik|z_1-z_2|] \dots \exp[ik|z_{n-1}-z_n|] \exp[ikz_n] \cdot \\ &\quad \cdot \Delta\epsilon(z_1) \Delta\epsilon(z_2) \dots \Delta\epsilon(z_n) dz_1 dz_2 \dots dz_n, \end{aligned} \right.$$

respectively. We remind the reader that T and R are random variables, a fact which may be emphasized by writing $T(\alpha)$ and $R(\alpha)$, *i.e.*, by explicitly indicating the realization index α . Since the random function $\Delta\epsilon(z)$ is constant in each slab, the multiple integrals appearing in (6) and (7) reduce to sums. Thus, using (1), we can write (6) as

$$(8) \quad T(\alpha) = 1 + \sum_{n=1}^{\infty} \left(\frac{k\Delta\epsilon}{2} \right)^n \sum_{r_1, r_2, \dots, r_n=1}^N a(r_1, r_2, \dots, r_n) \beta(r_1, \alpha) \beta(r_2, \alpha) \dots \beta(r_n, \alpha),$$

where the coefficients $a(r_1, r_2, \dots, r_n)$ are defined by

$$(9) \quad a(r_1, r_2, \dots, r_n) = i^n \int_{(r_1-1)d}^{r_1 d} \int_{(r_2-1)d}^{r_2 d} \dots \int_{(r_n-1)d}^{r_n d} \exp[-ikz_1] \exp[ik|z_1 - z_2|] \dots \cdot \exp[ik|z_n - z_{n-1}|] \exp[ikz_n] dz_1 dz_2 \dots dz_n.$$

The summation with respect to the indices r_1, r_2, \dots, r_n extends over all the N^n distinct n -tuples (r_1, r_2, \dots, r_n) . Similarly, we can write (7) as

$$(10) \quad R(\alpha) = \sum_{r=1}^{\infty} \left(\frac{k\Delta\epsilon}{2} \right)^n \sum_{r_1, r_2, \dots, r_n=1}^N b(r_1, r_2, \dots, r_n) \beta(r_1, \alpha) \beta(r_2, \alpha) \dots \beta(r_n, \alpha),$$

where the coefficients $b(r_1, r_2, \dots, r_n)$ are defined by

$$(11) \quad b(r_1, r_2, \dots, r_n) = i^n \int_{(r_1-1)d}^{r_1 d} \int_{(r_2-1)d}^{r_2 d} \dots \int_{(r_n-1)d}^{r_n d} \exp[ikz_1] \exp[ik|z_1 - z_2|] \dots \cdot \exp[ik|z_{n-1} - z_n|] \exp[ikz_n] dz_1 dz_2 \dots dz_n.$$

It is important to note that

$$(12) \quad |a(r_1, r_2, \dots, r_n)| \leq d^n, \quad |b(r_1, r_2, \dots, r_n)| \leq d^n,$$

which follows at once from the fact that the integrands in (9) and (11) are complex quantities with unit modulus.

Turning to the question of the convergence of the series (8) and (10), we note the inequalities

$$|T(\alpha)| \leq 1 + \sum_{n=1}^{\infty} \left(\frac{k d \Delta\epsilon}{2} \right)^n \sum_{r_1, \dots, r_n} |\beta(r_1, \alpha) \dots \beta(r_n, \alpha)| = 1 + \sum_{n=1}^{\infty} \left(\frac{k N d \Delta\epsilon}{2} \right)^n,$$

$$|R(\alpha)| \leq \sum_{n=1}^{\infty} \left(\frac{k d \Delta\epsilon}{2} \right)^n \sum_{r_1, \dots, r_n} |\beta(r_1, \alpha) \dots \beta(r_n, \alpha)| = \sum_{n=1}^{\infty} \left(\frac{k N d \Delta\epsilon}{2} \right)^n,$$

where we have used (12). It follows at once that the Neumann series (8) and (10) are absolutely convergent if

$$(13) \quad A = \frac{k d \Delta \varepsilon}{2} < \frac{1}{N}.$$

It should be noted that the convergence condition (13) guarantees the convergence of the Neumann series for $T(\alpha)$ and $R(\alpha)$ for all α in S .

In the remainder of this paper we shall be concerned with the mean square values of $T(\alpha)$ and $R(\alpha)$, since these are the quantities of greatest practical interest. Therefore, we now convert the Neumann series (8) and (10) into corresponding Neumann series for $\langle |T|^2 \rangle$ and $\langle |R|^2 \rangle$. By the angular brackets we mean the *ensemble average*, i.e., the average over all realizations of $\Delta \varepsilon(z)$. More explicitly, if $f(\alpha)$ is a function of the point α of the (discrete) sample space S , then

$$\langle f(\alpha) \rangle = \sum_{\alpha \text{ in } S} p(\alpha) f(\alpha),$$

where $p(\alpha)$ is the probability of the event corresponding to α ; in our case, S is the space consisting of the 2^N possible outcomes of a sequence of N independent Bernoulli trials, and $p(\alpha)$ has the value 2^{-N} for every α in S . Thus, multiplying (8) and (10) by their complex conjugates and ensemble averaging, we find

$$(14) \quad \langle |T|^2 \rangle = 1 + \sum_{n=1}^{\infty} \left(\frac{k \Delta \varepsilon}{2} \right)^{2n} \sum_{r_1, \dots, r_{2n}} [a(r_1, \dots, r_{2n}) + a^*(r_1, \dots, r_{2n})] \langle \beta(r_1) \dots \beta(r_{2n}) \rangle + \\ + \sum_{n=1}^{\infty} \left(\frac{k \Delta \varepsilon}{2} \right)^{2n} \sum_{\substack{l+m=2n \\ l, m > 0}} \sum_{r_1, \dots, r_l} \sum_{s_1, \dots, s_m} a(r_1, \dots, r_l) a^*(s_1, \dots, s_m) \langle \beta(r_1) \dots \beta(r_l) \beta(s_1) \dots \beta(s_m) \rangle,$$

and

$$(15) \quad \langle |R|^2 \rangle = \sum_{n=1}^{\infty} \left(\frac{k \Delta \varepsilon}{2} \right)^{2n} \sum_{\substack{l+m=2n \\ l, m > 0}} \sum_{r_1, \dots, r_l} \sum_{s_1, \dots, s_m} b(r_1, \dots, r_l) b^*(s_1, \dots, s_m) \cdot \\ \cdot \langle \beta(r_1) \dots \beta(r_l) \beta(s_1) \dots \beta(s_m) \rangle,$$

where we suppress the index α , and make use of the fact that averages of products of an odd number of the random variables $\beta(r)$ vanish. It is easily verified that the sums of the terms in (14) or (15) which are of a given order in $\Delta \varepsilon$ are real. Since

$$(16) \quad |\langle \beta(r_1) \dots \beta(r_l) \beta(s_1) \dots \beta(s_m) \rangle| \leq 1,$$

we have the inequalities

$$(17) \quad \langle |T|^2 \rangle \leq 1 + \sum_{n=1}^{\infty} (2n+1) \left(\frac{kNd\Delta\varepsilon}{2} \right)^{2n},$$

and

$$(18) \quad \langle |R|^2 \rangle \leq \sum_{n=1}^{\infty} (2n-1) \left(\frac{kNd\Delta\varepsilon}{2} \right)^{2n}.$$

Just as one would expect, these series converge if and only if the condition (13) is satisfied.

We shall now show that, *because of the random nature of the scattering medium* the convergence of the series (14) and (15) is usually more rapid than one would infer from the convergence condition (13). To show this, we must first prove an auxiliary inequality.

Basic inequality. — Let $c(r_1, r_2, \dots, r_n)$ stand for either the coefficient $a(r_1, r_2, \dots, r_n)$ defined by (9) or the coefficient $b(r_1, r_2, \dots, r_n)$ defined by (11). Let $\beta(r)$, $1 \leq r \leq N$, denote N independent random variables, each of which takes the values -1 and $+1$ with equal probability. Then, if $l, m > 0$ and $l+m=2n$,

$$(19) \quad \left| \sum_{r_1, \dots, r_l} \sum_{s_1, \dots, s_m} c(r_1, \dots, r_l) c^*(s_1, \dots, s_m) \langle \beta(r_1) \dots \beta(r_l) \beta(s_1) \dots \beta(s_m) \rangle \right| \leq d^n \left\langle \left[\sum_{r=1}^N \beta(r) \right]^{2n} \right\rangle.$$

Proof. — Writing the quantity

$$(20) \quad \langle \beta(r_1) \dots \beta(r_l) \beta(s_1) \dots \beta(s_m) \rangle$$

in the form $\langle \beta^{e_1}(1) \beta^{e_2}(2) \dots \beta^{e_N}(N) \rangle$, where e_1, e_2, \dots, e_N are suitable non-negative integers with sum $2n$, we see that (20) is non-negative, since it either vanishes (if any of the e_i is odd) or is positive (if none of the e_i is odd). Moreover, by (12), the coefficients $c(r_1, \dots, r_l) c^*(s_1, \dots, s_m)$ appearing in (19) all have absolute values $\leq d^{2n}$. The inequality (19) then follows from the obvious fact that if A, B, \dots, X are non-negative and if $|a|, |b|, \dots, |x| \leq C$, then

$$|aA + bB + \dots + xX| \leq C(A + B + \dots + X),$$

and from the identity

$$\sum_{r_1, \dots, r_l} \sum_{s_1, \dots, s_m} \langle \beta(r_1) \dots \beta(r_l) \beta(s_1) \dots \beta(s_m) \rangle = \left\langle \left[\sum_{r=1}^N \beta(r) \right]^{2n} \right\rangle, \quad l+m=2n.$$

Finally, we note that the inequality

$$(21) \quad \left| \sum_{r_1, \dots, r_{2n}} a(r_1, \dots, r_{2n}) \langle \beta(r_1) \dots \beta(r_{2n}) \rangle \right| \leq d^{2n} \left\langle \left[\sum_{r=1}^N \beta(r) \right]^{2n} \right\rangle$$

can be proved in just the same way.

3. - The stochastic majorants.

The qualitative idea behind the considerations of this section is the following: As shown by (13), the requirement $AN = kNd\Delta\epsilon/2 < 1$ is sufficient to guarantee the convergence of the Neumann series (14) and (15) for $\langle |T|^2 \rangle$ and $\langle |R|^2 \rangle$. On the other hand, it is clear that AN is not an appropriate quantity to use in estimating the *rate of convergence* of the series (14) and (15). To see this, we observe that we have not taken into account the random nature of the scattering medium in deriving (17) and (18), except, of course, insofar as we have used the fact that expectations of products of an odd number of $\beta(r)$ vanish. Indeed, by writing

$$(16) \quad |\langle \beta(r_1) \dots \beta(r_l) \beta(s_1) \dots \beta(s_m) \rangle| \leq 1,$$

we have automatically eliminated any effect that the statistics of the problem might have in attenuating high order scattering. Since, when N is large, most of the realizations of $\Delta\epsilon(z)$ exhibit many changes in sign, one might expect, to take a rather extreme example, that the internal field produced in the stack of slabs by single scattering is less than what it would be if the profile had the constant value $\Delta\epsilon$ throughout the stack of slabs (*). The rather loose phrases «coherent scattering» and «incoherent scattering» are ordinarily used to distinguish between these two situations. We shall now apply the lemmas of the preceding section to derive majorant series for the absolute value of the error committed in omitting terms of order $\geq 2n_0$ in the series (14) and (15) for $\langle |T|^2 \rangle$ and $\langle |R|^2 \rangle$. These majorants take into account the random nature of the scattering medium, *i.e.*, the oscillatory nature or «wiggleness» of most of the realizations of $\Delta\epsilon(z)$, and will accordingly be called *stochastic majorants*. We shall find that, although the condition $AN < 1$ is still needed to ensure the convergence of the stochastic majorant series, a fact which is perhaps surprising at first, the rate of convergence of the stochastic majorants is usually much faster than would be inferred from the size of AN .

The derivation of the stochastic majorants follows almost immediately when, instead of the crude inequality (16), we use the more refined inequalities (19) and (21), proved in the preceding section. Thus, except for the factor $(k\Delta\epsilon/2)^{2n}$, the term of order $2n$ in the sum (14) is the sum of

$$(22) \quad \sum_{r_1, \dots, r_{2n}} [a(r_1, \dots, r_{2n}) + a^*(r_1, \dots, r_{2n})] \langle \beta(r_1) \dots \beta(r_{2n}) \rangle$$

(*) In our case, there is probability 2^{-N} of this occurring.

and

$$(23) \quad \sum_{\substack{l+m=2n \\ l, m > 0}} \sum_{r_1, \dots, r_l} \sum_{s_1, \dots, s_m} a(r_1, \dots, r_l) a^*(s_1, \dots, s_m) \langle \beta(r_1) \dots \beta(r_l) \beta(s_1) \dots \beta(s_m) \rangle.$$

Using (21), we see that the absolute value of (22) is bounded above by

$$2d^{2n} \langle [\sum_{r=1}^N \beta(r)]^{2n} \rangle.$$

Moreover, using (19), we see that the absolute value of (23) is bounded above by

$$(2n-1)d^{2n} \langle [\sum_{r=1}^N \beta(r)]^{2n} \rangle.$$

It follows at once from the form of (14) that if $E_T(2n_0)$ is the error committed in neglecting terms in (14) with orders $\geq 2n_0$, then

$$(24) \quad |E_T(2n_0)| \leq \sum_{n=n_0}^{\infty} (2n+1) \left(\frac{kd\Delta\varepsilon}{2} \right)^{2n} \langle \beta_N^{2n} \rangle, \quad 0 \leq n_0 < \infty,$$

where the zero-mean random variable β_N is defined by

$$(25) \quad \beta_N = \sum_{r=1}^N \beta(r),$$

i.e., β_N is the sum of N independent random variables, each of which takes on the values -1 and 1 with equal probability. The distribution of β_N is, of course, the familiar binomial distribution of probability theory, for which

$$(26) \quad \text{Prob} \{ \beta_N = N - 2r \} = 2^{-N} \binom{N}{r}, \quad 0 \leq r \leq N.$$

The difference between the right hand side of (24) and the corresponding terms of (17) is that we have

$$(27) \quad \langle \beta_N^{2n} \rangle = 2^{-N} \sum_{r=0}^N \binom{N}{r} (N-2r)^{2n},$$

in place of N^{2n} . Thus, the crudeness of the estimate (17) consists in replacing $\langle \beta_N^{2n} \rangle$ by the value N^{2n} , i.e., in failing to take account of the fact that the random variable β_N^{2n} takes on its maximum value of N^{2n} only with the probability 2^{1-N} , which is small when the number of slabs is large.

In just the same way, using (15), (19) and (21), we find that if $E_R(2n_0)$ is the error committed in neglecting terms in (15) of order $\geq 2n_0$, then

$$(28) \quad |E_R(2n_0)| \leq \sum_{n=n_0}^{\infty} (2n-1) \left(\frac{kd\Delta\varepsilon}{2} \right)^{2n} \langle \beta_N^{2n} \rangle, \quad 1 \leq n_0 < \infty,$$

which again differs from the corresponding terms of (18) only in the important respect, of having $\langle \beta_N^{2n} \rangle$ in place of N^{2n} . Since the majorant series (17) and (18) do not take account of the random nature of the scattering medium, we shall refer to them for brevity as the *sure* majorants, using the word *sure* as an antonym for *stochastic*.

4. - The binomial moments.

To proceed with the analysis of the stochastic majorants, we must first study the behavior of the binomial moment $\langle \beta_N^{2n} \rangle$ as a function of N , the number of slabs, and $2n$, the order of the moment (the odd moments vanish). For convenience, we introduce the *scaled* random variable.

$$\hat{\beta}_N = A\beta_N,$$

and deal with the corresponding *scaled* binomial moment

$$\langle \hat{\beta}_N^{2n} \rangle = \langle (A\beta_N)^{2n} \rangle = \left\langle \left(\frac{k\bar{d}\Delta\epsilon}{2} \beta_N \right)^{2n} \right\rangle,$$

rather than $\langle \beta_N^{2n} \rangle$. It follows from $\langle (A\beta_N)^{2n} \rangle \leq (AN)^{2n}$ and $AN < 1$ that $\langle \hat{\beta}_N^{2n} \rangle$ approaches zero with increasing n , unlike $\langle \beta_N^{2n} \rangle$. By (27), we have

$$(29) \quad \langle \hat{\beta}_N^{2n} \rangle = 2^{-N} \sum_{r=0}^N \binom{N}{r} [A(N-2r)]^{2n},$$

obviously a monotone decreasing function of $2n$. It is to be expected that the behavior of $\langle \hat{\beta}_N^{2n} \rangle$ depends on the relative size of $2n$ and N .

Qualitatively, the conclusions which we shall establish in this section are the following: When $2n \ll N$, $\langle \hat{\beta}_N^{2n} \rangle$ can be estimated by the corresponding (scaled) moment $\langle g_N^{2n} \rangle$ of g_N , the zero-mean Gaussian random variable with the same variance; this is reasonable in view of the central limit theorem of probability theory. As $2n$ approaches N , i.e., as the order of the moment approaches the number of slabs, this approximation breaks down completely, and we must use (29) to find $\langle \hat{\beta}_N^{2n} \rangle$. Indeed, for fixed N , the quantity $\langle g_N^{2n} \rangle$ instead of going to zero becomes infinite as n increases. When $2n \gg N$, the scaled binomial moment $\langle \hat{\beta}_N^{2n} \rangle$ can be estimated by the quantity $2^{1-N}(AN)^{2n}$, which goes to zero exponentially as n increases. However, as is apparent from this exponential estimate, the condition $AN < 1$ is still needed to ensure the convergence even of the stochastic majorants.

We now discuss these matters in more detail.

4.1. *The Gaussian estimate of $\langle \hat{\beta}_N^{2n} \rangle$.* — To obtain the Gaussian estimate of $\langle \hat{\beta}_N^{2n} \rangle$, we consider the zero-mean Gaussian random variable g_N with the same variance as $\hat{\beta}_N$, i.e., with variance $A^2 N$. The Gaussian moments are given by ⁽²⁾

$$(30) \quad \langle g_N^{2n} \rangle = 1 \cdot 3 \dots (2n-1) A^{2n} N^n,$$

while all odd moments of g_N vanish. Moreover, we have the inequality

$$(31) \quad \langle \hat{\beta}_N^{2n} \rangle \leq \langle g_N^{2n} \rangle, \quad 1 \leq N < \infty, \quad 0 \leq n < \infty,$$

which means that the Gaussian moments are upper bounds for the binomial moments. To show this, we first observe that

$$\hat{\beta}_N = \sum_{r=1}^N \hat{\beta}(r),$$

where $\hat{\beta}(r)$, $1 \leq r \leq N$, is a sequence of independent Bernoulli random variables, each of which takes the values $-A$ and $+A$ with equal probability. We then write

$$g_N = \sum_{r=1}^N g(r),$$

where $g(r)$, $1 \leq r \leq N$, is a sequence of independent Gaussian random variables, each with mean zero and variance A^2 . It follows by the multinomial expansion that

$$(32) \quad \langle \hat{\beta}_N^{2n} \rangle = \sum_{m_1, \dots, m_N} \frac{(2n)!}{m_1! \dots m_N!} \langle \hat{\beta}^{m_1}(1) \rangle \dots \langle \hat{\beta}^{m_N}(N) \rangle,$$

$$(33) \quad \langle g_N^{2n} \rangle = \sum_{m_1, \dots, m_N} \frac{(2n)!}{m_1! \dots m_N!} \langle g^{m_1}(1) \rangle \dots \langle g^{m_N}(N) \rangle,$$

where the m_r , $1 \leq r \leq N$, are arbitrary non-negative integers subject to the condition $m_1 + m_2 + \dots + m_N = 2n$. Since all the moments of $\hat{\beta}(r)$, $g(r)$, $1 \leq r \leq N$, are non-negative, the desired inequality (31) follows at once from a term-by-term comparison of (32) and (33), together with the obvious inequalities

$$\langle \hat{\beta}^{2n} \rangle = A^{2n} \leq 1 \cdot 3 \dots (2n-1) A^{2n} = \langle g^{2n} \rangle,$$

$$\langle \hat{\beta}^{2n+1} \rangle = 0 \leq 0 = \langle g^{2n+1} \rangle, \quad 0 \leq n < \infty,$$

where $\hat{\beta}$ denotes any of the $\hat{\beta}(r)$ and g denotes any of the $g(r)$.

⁽²⁾ H. CRAMÉR: *Mathematical methods of statistics* (Princeton, 1946), p. 212.

Applying the DeMoivre-Laplace limit theorem ⁽³⁾ (the special case of the central theorem appropriate to a sum of independent, identically distributed Bernoulli random variables), we would expect to find that $\hat{\beta}_N$ is approximately Gaussian, with mean zero and variance A^2N , to an approximation which improves as N increases. The difference between the distribution of $\hat{\beta}_N$ and that of g_N is greatest in the tails of the distributions; in particular, the values of $\hat{\beta}_N$ are confined to the interval $(-AN, +AN)$, whereas g_N takes on values outside any finite interval. Thus, for fixed N , we can expect $\langle g_N^{2n} \rangle$ to become a poorer approximation to $\langle \hat{\beta}_N^{2n} \rangle$ as n increases. In fact, for large n we can use Stirling's formula

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} \exp[-n]$$

to derive the approximation

$$(34) \quad \langle g_N^{2n} \rangle \sim \sqrt{2} \left(\frac{2nA^2N}{e} \right)^n.$$

It follows that ultimately $\langle g_N^{2n} \rangle$ goes to infinity as n increases, whereas, as already noted, $\langle \hat{\beta}_N^{2n} \rangle$ must go to zero as n increases. (We assume, of course, that the condition $AN < 1$ is satisfied).

Using (34), we easily see that the minimum value of $\langle g_N^{2n} \rangle$ is obtained for

$$(35) \quad 2n_1 \sim \frac{1}{A^2N},$$

and is approximately $\sqrt{2} \exp[-n_1]$.

These properties of the Gaussian estimate of $\langle \hat{\beta}_N^{2n} \rangle$ are illustrated in Fig. 2, where we have plotted $\langle \hat{\beta}_N^{2n} \rangle$ and $\langle g_N^{2n} \rangle$ for the case $N = 20$, $AN = 0.99$. It

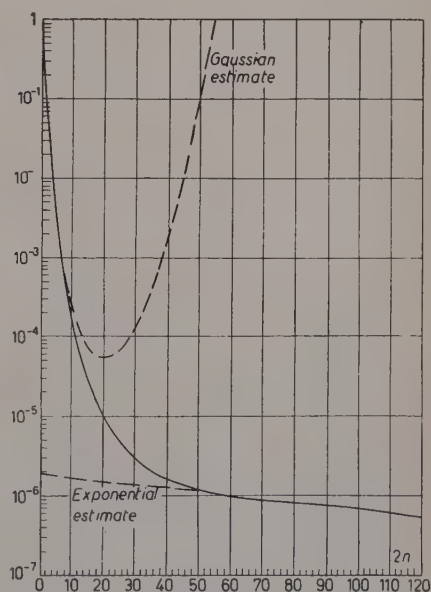


Fig. 2. — The solid curve represents the scaled binomial moment $\langle \hat{\beta}_N^{2n} \rangle$ as a function of $2n$, for $N = 20$ and $AN = 0.99$. The dashed curves give the Gaussian and exponential estimates of $\langle \hat{\beta}_N^{2n} \rangle$.

⁽³⁾ W. FELLER: *An introduction to probability theory and its applications* (2nd edition), (New York, 1957), p. 172.

will be noted that the Gaussian estimate of $\langle \hat{\beta}_N^{2n} \rangle$ is good for $n \leq 10$, and then deteriorates rapidly. The quantity $\langle g_N^{2n} \rangle$ is a minimum for $2n_1 = 20 \sim (A^2 N)^{-1}$, in agreement with (35); it then increases rapidly and ultimately diverges.

4.2. *The exponential estimate of $\langle \hat{\beta}_N^{2n} \rangle$.* — When $2n \gg N$, we have the approximation

$$\langle \hat{\beta}_N^{2n} \rangle \sim 2^{1-N} (AN)^{2n},$$

which improves as n increases. To see this, we examine the terms of the sum (29) in the limit as $n \rightarrow \infty$. Thus, for example, the ratio of the tail term corresponding to $r=0$ to the term corresponding to $r=1$ is

$$\frac{1}{N} \left(\frac{N}{N-2} \right)^{2n},$$

which goes to infinity as $n \rightarrow \infty$. Applying this argument to the other terms in the sum (29), it is clear that as n increases, most of the contribution to $\langle \hat{\beta}_N^{2n} \rangle$ comes from the two tail terms corresponding to $r=0$ and $r=N$, which in the limit of infinite n completely dominate the other terms of the sum (29). It follows that for large n

$$\langle \hat{\beta}_N^{2n} \rangle \sim 2^{-N} \binom{N}{0} (AN)^{2n} + 2^{-N} \binom{N}{N} (-AN)^{2n} = 2^{1-N} (AN)^{2n},$$

as asserted. In particular, the condition $AN < 1$ is still needed to ensure the convergence of the stochastic majorants.

The exponential estimate $2^{1-N} (AN)^{2n}$ is drawn in Fig. 2 for the case $N=20$, $AN=0.99$. We see that in this case $\langle \hat{\beta}_N^{2n} \rangle$ becomes asymptotic to its exponential estimate for $2n \gtrsim 70$. The portion of the $\langle \hat{\beta}_N^{2n} \rangle$ vs. $2n$ curve for which neither the Gaussian estimate nor the exponential estimate is good must be filled in by using (29).

Tables I and II give values of $\langle \hat{\beta}_N^{2n} \rangle$, in the ranges where they are appreciable, for $N=20, 35, 50$ and $AN=0.50, 0.99$. The use of such tables in estimating the strength of multiple scattering is discussed in the next section.

TABLE I. — *The scaled binomial moments $\langle \hat{\beta}_N^{2n} \rangle$ for the case $AN = 0.99$ and various values of N . The numbers in parentheses are the corresponding Gaussian estimates of $\langle \hat{\beta}_N^{2n} \rangle$.*

$2n$	$N = 20$	$N = 35$	$N = 50$	$2n$	$N = 20$
2	.049005000 (.049005000)	.028002857 (.028002857)	.019602000 (.019602000)	38	.000001830
4	.006964321 (.007204470)	.002307671 (.002352480)	.001137346 (.001152715)	40	.000001686
6	.001593455 (.001765275)	.000310846 (.000329381)	.000108507 (.000112978)	42	.000001571
8	.000492723 (.000605551)	.000057478 (.000064565)	.000014296 (.000015502)	44	.000001476
10	.000188968 (.000267075)	.000013395 (.000016272)	.000002389 (.000002735)	46	.000001398
12	.000085391 (.000143968)	.000003740 (.000005012)	.000000481 (.000000590)	48	.000001332
14	.000043934 (.000091717)	.000001209 (.000001825)	.000000113 (.000000150)	50	.000001276
16	.000025112 (.000067419)	.000000442 (.000000766)	.000000030 (.000000044)	52	.000001227
18	.000015654 (.000056166)	.000000179 (.000000365)	.000000009 (.000000015)	54	.000001185
20	.000010490 (.000052296)	.000000080 (.000000019)	.000000003 (.000000005)	56	.000001147
22	.000007469	.000000038	.000000001	58	.000001113
24	.000005597	.000000020	—	60	.000001081
26	.000004380	.000000011	less	62	.000001053
28	.000003555	.000000006	than	64	.000001026
30	.000002977	.000000004	10^{-9}	66	.000001001
32	.000002559	.000000002	—	68	.000000978
34	.000002248	.000000002	—	70	.000000956
36	.000002013	.000000001	—	72	.000000934

TABLE II. — *The scaled binomial moments $\langle \hat{\beta}_N^{2n} \rangle$ for the case $AN=0.50$ and various values of N . The numbers in parentheses are the corresponding Gaussian estimates of $\langle \hat{\beta}_N^{2n} \rangle$*

$2n$	$N=20$	$N=35$	$N=50$
2	.012500000 (.012500000)	.007142857 (.007142857)	.005000000 (.005000000)
4	.000453125 (.000468750)	.000150146 (.000153061)	.000074000 (.000075000)
6	.000026445 (.000029297)	.000005159 (.000005466)	.000001801 (.000001875)
8	.000002086 (.000002563)	.000000243 (.000000273)	.000000061 (.000000066)
10	.000000204 (.000000288)	.000000014 (.000000018)	.000000003 (.000000003)
12	.000000024 (.000000040)	.000000001 (.000000001)	less than 10^{-9}
14	.000000003 (.000000006)	less than 10^{-9}	—

5. — Analysis of multiple scattering.

The application of the results obtained in the preceding section to the problem of multiple scattering in a binomially distributed, random stack of dielectric slabs is straightforward since the terms of the stochastic majorant series (24) and (28) are just the binomial moments $\langle \hat{\beta}_N^{2n} \rangle$ multiplied by the growth factors $2n+1$ and $2n-1$ (*). To illustrate the situation, we plot in Fig. 3 the general term $(2n-1)\langle \hat{\beta}_N^{2n} \rangle$ of the stochastic majorant series (28) for $|E_R(2n_0)|$, for the case $AN=0.99$, $N=20$. Comparing Figs. 2 and 3, we see that in both cases the terms at first fall off rapidly with increasing n , then fall off less rapidly as $2n$ approaches N , and finally approach an asymptotic curve corresponding to the exponential estimate of $\langle \hat{\beta}_N^{2n} \rangle$. The exponential

(*) These factors represent the number of terms in the Neumann series for $T(\alpha)$ and $R(\alpha)$ which combine to give the term of order $2n$ in the Neumann series for $\langle |R|^2 \rangle$ and $\langle |T|^2 \rangle$.

estimate of the general term of the stochastic majorant series (24) is given by

$$(36) \quad (2n-1) \langle \hat{\beta}_N^{2n} \rangle \sim (2n-1) 2^{1-N} (AN)^{2n}.$$

The quantity (36) is a maximum for

$$2n-1 \sim -\frac{1}{\log_e AN}.$$

For the case $AN = 0.99$, this maximum occurs at $2n-1 \sim 100$ (see Fig. 3).

For comparison, we also plot in Fig. 3 the terms of the sure majorant series (18), which, it will be recalled, differ from the corresponding terms of the stochastic majorant series (28) by having $(AN)^{2n}$ in place of $\langle \hat{\beta}_N^{2n} \rangle$. Thus, except for the factor 2^{1-N} , the exponential estimate of the terms of the stochastic majorant series is the same as for the sure majorant series. This is apparent from Fig. 3, where the curves representing the terms of the sure majorant series and the exponential estimate of the terms of the stochastic majorant series differ only by a vertical translation. The appearance of the terms of the stochastic majorant series (24) for $|E_T(2n_0)|$ differs only slightly from that of the stochastic majorant series (28) for $|E_R(2n_0)|$, due to the closeness of the growth factors $2n-1$ and $2n+1$ for large n .

It is now a simple matter to estimate the error committed in neglecting multiple scattering greater than a given order. We illustrate the general procedure by two examples.

5.1. *Mean square reflection coefficient:* $AN = 0.99$, $N = 20$. - As we have seen in Sect. 4, the absolute value of the error $E_R(2n_0)$ committed by neglecting the terms of order $2n_0$ and greater in the Neumann series (15) for $\langle |R|^2 \rangle$ is bounded above by the sum

$$(37) \quad \sum_{n=n_0}^{\infty} (2n-1) \langle \hat{\beta}_N^{2n} \rangle.$$

To evaluate (37) for various values of n_0 , we use the data of Table II, to-

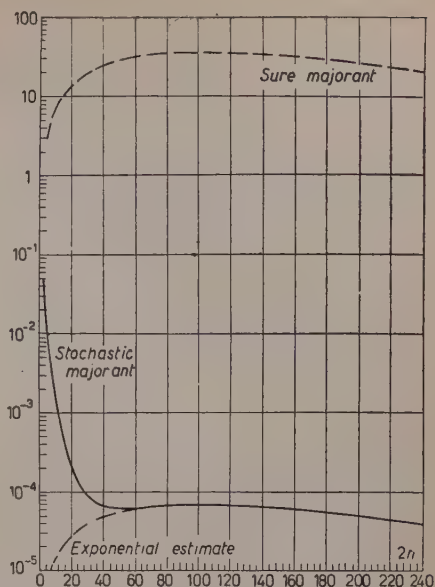


Fig. 3. - The solid curve represents the stochastic majorant series (28), when $N = 20$ and $AN = 0.99$. The dashed curves represent the sure majorant series and the exponential estimate of the stochastic majorant series.

gether with the estimate

(38)
$$\sum_{n=35}^{\infty} (2n-1) \langle \hat{\beta}_N^{2n} \rangle \sim 2^{1-N} \sum_{n=35}^{\infty} (2n-1) (AN)^{2n}.$$

The use of (38) is justified by the accuracy of the exponential estimate of $\langle \hat{\beta}_N^{2n} \rangle$ for $2n \gtrsim 70$. It is not hard to see that the right hand side of (38) equals

$$2^{1-N} (AN)^{70} \frac{69 - 67(AN)^2}{(1 - (AN)^2)^2} = .000794,$$

whence $|E_R(2n_0)|$ is bounded above by (*)

(39)
$$B(2n_0) \sim \sum_{n=n_0}^{34} (2n-1) \langle \hat{\beta}_N^{2n} \rangle + .000794.$$

The data presented in Table III were obtained by using (39).

TABLE III. - $B(2n_0)$ is an upper bound for the absolute value of the error committed by neglecting terms of order $\geq 2n_0$ in the Neumann series (15) for $\langle |R|^2 \rangle$, for the case $AN=0.99$, $N=20$.

$2n_0$	$B(2n_0)$	$2n_0$	$B(2n_0)$
4	.039	20	.003
6	.018	30	.002
8	.010	40	.002
10	.007	50	.001

It is easy to calculate explicitly the first term R_2 of the Neumann series (15) for $\langle |R|^2 \rangle$, since it is just the mean square absolute value of the random variable

(40)
$$\frac{ik}{2} \int_0^{Nd} \exp [2ikz] \Delta \varepsilon(z) dz.$$

The integral in (40) reduces to

$$\sum_{r=1}^N \Delta \varepsilon(r) \int_{(r-1)d}^{rd} \exp [2ikz] dz = \frac{\sin kd}{k} \sum_{r=1}^N \Delta \varepsilon(r) \exp [ik(r-1)d],$$

(*) To ensure that (39) is an upper bound, one could multiply the second term in the right hand side of (39) by 1.1, say. This would not affect the data of Table III to the accuracy given.

whence

$$(41) \quad R_2 = A^2 N \left(\frac{\sin kd}{kd} \right)^2.$$

On the other hand, the stochastic majorant for R_2 is, of course, the quantity $A^2 N$. It follows that R_2 is close to its stochastic majorant if $kd \ll 1$, *i.e.*, if the slab width d is small compared to the wavelength of the incident radiation. Since in the present case $A^2 N = 0.049005$, Table III shows that the first term in (15) and the absolute value of the sum of the next few terms may well be of comparable size. Indeed, if kd is large, we cannot even guarantee that single scattering predominates.

5.2. *Mean square transmission coefficient*; $AN = 0.50$, $N = 250$, $A^2 N = 10^{-3}$.
 — We begin by showing that the second term T_2 of the Neumann series (14) for $\langle |T|^2 \rangle$ is non-positive. It follows from (6) that T_2 is the sum of the two terms

$$(42) \quad \frac{k^2}{4} \left\langle \left| \int_0^{Nd} \Delta \varepsilon(z) dz \right|^2 \right\rangle,$$

and

$$(43) \quad -\frac{k^2}{4} \left\langle \int_0^{Nd} \int_0^{Nd} \exp[-ikz_1] \exp[ik|z_1 - z_2|] \exp[ikz_2] \Delta \varepsilon(z_1) \Delta \varepsilon(z_2) dz_1 dz_2 \right\rangle \\ + \text{complex conjugate}.$$

Since

$$\int_0^{Nd} \Delta \varepsilon(z) dz = d \sum_{r=1}^N \Delta \varepsilon(r),$$

(43) reduces to

$$(44) \quad \frac{k^2 d^2}{4} \left\langle \sum_{r=1}^N \Delta \varepsilon(r) \sum_{s=1}^N \Delta \varepsilon(s) \right\rangle = A^2 N.$$

The integral in (43) need only be evaluated for the case where z_1 and z_2 both lie in the same interval from $(r-1)d$ to rd , since otherwise the integral vanishes. Doing the integration, we find that (43) reduces to

$$(45) \quad -\frac{k^2}{4} \left[\frac{d^2}{2} - \frac{d}{2ik} - \frac{1}{4k^2} (\exp[2ikd] - 1) \right] \left\langle \sum_{r=1}^N (\Delta \varepsilon(r))^2 \right\rangle + \text{complex conjugate} = \\ = -A^2 N - A^2 N \left(\frac{\sin kd}{kd} \right)^2.$$

Combining (44) and (45), we obtain

$$T_2 = -A^2N \left(\frac{\sin kd}{kd} \right)^2,$$

which is non-positive. The quantity T_2 is bounded in absolute value by its stochastic majorant A^2N , and is approximately $-A^2N$ for small values of kd . Furthermore, we note that the sum of T_2 and the quantity R_2 given by (41) vanishes, a result which might be expected, since the sum of $\langle |T|^2 \rangle$ and $\langle |R|^2 \rangle$ must be unity by energy conservation.

We now estimate the error committed when we approximate the mean square transmission coefficient by

$$(46) \quad \langle |T|^2 \rangle \sim 1 + T_2 = 1 - A^2N \left(\frac{\sin kd}{kd} \right)^2 = 1 - 10^{-3} \left(\frac{\sin kd}{kd} \right)^2,$$

thereby neglecting all terms of order ≥ 4 in the Neumann series (14). To do this, we first sum the terms of order $\geq 2\nu$ in the *sure* majorant series (17) for $\langle |T|^2 \rangle$, obtaining

$$(47) \quad \sum_{n=\nu}^{\infty} (2n+1)(AN)^{2n} = \frac{(AN)^{2\nu}}{(1-(AN)^2)^2} [(2\nu+1) + (1-2\nu)(AN)^2].$$

For the value $AN=0.5$ under consideration, and for $\nu=15$, this sum is approximately 10^{-8} , which is an upper bound, albeit a very crude one, for the terms of order ≥ 30 , which were omitted in writing (46). The rest of the terms omitted in writing (46), with orders from 4 to 28, are bounded above by the series

$$(48) \quad \sum_{n=2}^{14} (2n+1) \langle g_N^{2n} \rangle,$$

i.e., by the Gaussian estimate of the corresponding terms of the *stochastic* majorant series (24), which, since $N=250$, should be quite a good estimate. Using (30), and then adding (47) and (48), we easily find that for the case $A^2N=10^{-3}$ the absolute value of the error committed in using the approximation (46) is less than $1.6 \cdot 10^{-5}$, say, which is only a small percentage of $|T_2|$ if $kd \ll 1$.

6. - Conclusion.

We have seen that the absolute values of successive terms in the Neumann series (14) and (15) for the mean square transmission coefficient and the mean square reflection coefficient may fall off rapidly with increasing n , even if the

convergence parameter AN is close to unity. Indeed, an examination of the formula (30) for the Gaussian estimates of the binomial moments shows that the smallness of $A\sqrt{N}$ (rather than of AN) is enough to ensure that the terms of the series (14) and (15) fall off rapidly in absolute value, at least until the order of the terms is comparable with the number of slabs N . In particular, if $kd < \pi$, then for sufficiently small $A\sqrt{N}$, the formulas

$$\langle |R|^2 \rangle \sim R_2, \quad \langle |T|^2 \rangle \sim 1 + T_2,$$

corresponding to the neglect of terms of order higher than two in the Neumann series (14) and (15), will be good approximations, even if AN is close to unity. (Note that $AN \sim 1$ and $A\sqrt{N} \ll 1$ imply that $N \gg 1$.) One might adduce this example in support of the general belief that in the problem of radio scattering beyond the horizon, the large number of «scattering blobs» within the common volume of the antenna beams helps to justify the conventional use of the single scattering approximation for the scattered field.

When the order of the terms is greater than N , we find that we must again use the size of AN , rather than the size of $A\sqrt{N}$, to estimate the rate of convergence of the series for $\langle |T|^2 \rangle$ and $\langle |R|^2 \rangle$. Qualitatively speaking, as n increases, the random nature of the scattering medium becomes progressively less effective in attenuating scattering of order n ; finally, when n becomes large compared to the number of slabs, the randomness of the medium plays no further role in attenuating multiple scattering.

A Monte-Carlo Method to Calculate Multiple Phase Space Integrals - I.

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CONTENTS. — 1. Introduction. — 2. Preparation of $\varrho_n^*(EP)$ for the application of MC. — 3. The MC approach. — 4. The computing programme. — 5. The spectrum. — Appendix.

1. — Introduction.

In the region of primary (lab) energies of the order of 1 to 100 GeV, the number of particles (mesons, heavy mesons, anti-nucleons, hyperons) created in a collision is of the order of 1 to 10. Consequently, pure thermodynamical calculations fail and if one wishes to apply a theory like Fermi's ⁽¹⁾ one has to calculate phase space integrals. We do not consider here the physical side of the problem, *i.e.* the validity of this type of theory. We only try to solve the mathematical problem of calculating the phase space integral, which has not been achieved in a simple and reliable manner so far. The integral, for n emerging particles with total energy E and total momentum P , is given by ⁽²⁾

$$(1) \quad \varrho_n(E, P) = (2\pi\hbar)^{-(3n-3)} \int d\mathbf{p}_1 \dots d\mathbf{p}_n \delta(\mathbf{P} - \sum_{i=1}^n \mathbf{p}_i) \delta(E - \sum_{i=1}^n \sqrt{p_i^2 + m_i^2}).$$

This integral does not depend on the direction of P . In the center-of-mass system (C.M.S.) we have $P = 0$. In what follows, we shall consider frequently

⁽¹⁾ E. FERMI: *Progr. Theor. Phys.*, **5**, 570 (1950).

⁽²⁾ The phase space integral is of course a function also of m_1, m_2, \dots, m_n . As a shorthand notation we write $\varrho_n(E, P)$ instead of $\varrho_{m_1 \dots m_n}(E, P)$.

instead of (1) the unnormalized phase space integral $\varrho_n^*(E, P)$

$$(2) \quad \varrho_n^*(E, P) = (2\pi\hbar)^{3n-3} \cdot \varrho_n(E, P).$$

The main difficulty in all attempts to perform the integrations have been the square roots $\sqrt{p_i^2 + m_i^2}$. The exact integration is possible only up to $n = 3$, for higher numbers of particles one has to integrate numerically. In principle this can be done by repeated convolution since

$$(3) \quad \varrho_n^*(E, P) = \int d\mathbf{p}_n \varrho_{n-1}^*(E - \sqrt{p_n^2 + m_n^2}, \mathbf{P} - \mathbf{p}_n)$$

and this is in fact the way BLOCK⁽³⁾ has done the first few integrations. But for larger numbers n the repeated numerical convolution becomes either very inaccurate or prohibitively long. Several authors have tried approximations⁽⁴⁾, but in many cases it is difficult to estimate the error. In unfavourable circumstances, the result can be wrong by a factor five, even if for other cases the error is only some 10%.

We may hope that by keeping the square root (instead of approximating it by either p_i or $p_i^2/2m_i$ as in most of the other calculations) but using a Monte-Carlo method (M.C.) for the integration, we may achieve an accuracy of a few percent. The method has the advantage that it gives in the course of calculating $\varrho_n^*(E, 0)$ automatically the energy spectra of each kind of particles.

Exact spectra and phase space densities should be useful in extracting from future experiments a suitable representation of the matrix elements of interaction by simple functions of energy, numbers of particles, etc. These functions (the «interaction volume» in the simplest case) may be considered as parameters of the theory whose order of magnitude and general behaviour can be concluded from physical arguments but which still remain somewhat arbitrary. We feel that such a theory is not very satisfactory, but it may, as a half-empirical procedure, lead to good qualitative predictions about the reactions of elementary particles between 1 ÷ 100 GeV. Questions of this kind have turned out to be interesting since accelerators are under construction, which will yield protons of 25 and more GeV.

In all that follows, we treat the particles as if they were distinguishable and consequently each one has a «label»: 1, 2, ..., n . In any application one has then to correct ϱ_n by suitable factors⁽⁵⁾.

(3) M. M. BLOCK: *Phys. Rev.*, **101**, 796 (1956).

(4) a) R. MILBURN: *Rev. Mod. Phys.*, **27**, 1 (1955); b) S. BELEN'KIJ *et al.*: *Usp. Fiz. Nauk*, **62**, 1 (1957); c) G. E. A. FIALHO: *Phys. Rev.*, **105**, 328 (1957) and various other papers quoted in a) and b).

(5) Y. YEIVIN and A. DE-SHALIT: *Nuovo Cimento*, **1**, 1146 (1956).

2. - Preparation of $\varrho_n^*(E, P)$ for the application of MC.

First of all, we do as much analytical evaluation of the integral as possible and apply MC only at the end.

We put

$$\mathbf{p}_i = p_i \mathbf{e}_i; \quad |\mathbf{e}_i| = 1$$

$$d\mathbf{p}_i \equiv p_i^2 dp_i d\mathbf{e}_i = p_i^2 \sin \theta_i d\theta_i d\varphi_i dp_i$$

and have

$$(4) \quad \varrho(E_n^*, P) =$$

$$= \int_0^\infty \dots \int_0^\infty dp_1 \dots dp_n p_1^2 \dots p_n^2 \delta(E - \sum_{i=1}^n \sqrt{p_i^2 + m_i^2}) \int_{\text{all directions}} \delta(\mathbf{P} - \sum_{i=1}^n p_i \mathbf{e}_i) d\mathbf{e}_1 \dots d\mathbf{e}_n.$$

In this way the whole integral is split into two parts.

The function

$$(5) \quad w_n(P, p_1 \dots p_n) = \left(\frac{1}{4\pi}\right)^n \int \delta(\mathbf{P} - \sum p_i \mathbf{e}_i) d\mathbf{e}_1 \dots d\mathbf{e}_n$$

has a very simple meaning: $w_n(P, p_1 \dots p_n) d\mathbf{P}$ is the probability that for given $p_1 \dots p_n$, but random directions $\mathbf{e}_1 \dots \mathbf{e}_n$ the resultant vector $\sum p_i \mathbf{e}_i$ lies in the neighbourhood $d\mathbf{P}$ of \mathbf{P} . It is normalized such that

$$(6) \quad \int w_n(P, p_1 \dots p_n) d\mathbf{P} = 4\pi \int_0^\infty P^2 w_n(P, p_1 \dots p_n) dP = 1,$$

and it depends in fact only on $|\mathbf{P}|$. We shall consequently refer to this function as the «random walk function». An explicit derivation of this result and some further properties of the function $w_n(P, p_1 \dots p_n)$ are found in the Appendix.

Introducing now the energies by

$$(7) \quad \varepsilon_i = \sqrt{p_i^2 + m_i^2}, \quad p_i(\varepsilon_i) = \sqrt{\varepsilon_i^2 - m_i^2},$$

and defining

$$(8) \quad u_i(\varepsilon) = \begin{cases} \varepsilon \sqrt{\varepsilon^2 - m_i^2} & \varepsilon > m_i \\ 0 & \varepsilon \leq m_i \end{cases}; \quad v_n(P, \varepsilon_1 \dots \varepsilon_n) = w_n(P, p_1 \dots p_n),$$

we may write (4) in the center of mass system (CMS) as

$$(9) \quad \varrho_n^*(E, 0) = (4\pi)^n \int_{m_1}^\infty d\varepsilon_1 \dots \int_{m_n}^\infty d\varepsilon_n u_1(\varepsilon_1) \dots u_n(\varepsilon_n) v_n(0, \varepsilon_1 \dots \varepsilon_n) \delta(E - \sum \varepsilon_i).$$

A further transformation from the total energies to the kinetic energies

$$(10) \quad t_i = \varepsilon_i - m_i; \quad \sum m_i = M; \quad E - M = T$$

gives

$$(11) \quad \begin{cases} \varrho_n^*(E, 0) = (4\pi)^n \int_0^\infty dt_1 \dots \int_0^\infty dt_n \Phi(t_1 \dots t_n) \delta(T - \sum t_i), \\ \Phi(t_1 \dots t_n) = u_1(t_1 + m_1) \dots u_n(t_n + m_n) \cdot v_n(0, t_1 + m_1, \dots, t_n + m_n). \end{cases}$$

A last transformation gives the final form to which MC will be applied:

$$(12) \quad \begin{cases} t_1 = T_1 \\ t_2 = T_2 - T_1 \\ t_n = T_n - T_{n-1} \end{cases}$$

leads to

$$\varrho_n^*(E, 0) = (4\pi)^n \int_0^\infty dT_1 \int_{T_1}^\infty dT_2 \dots \int_{T_{n-2}}^\infty dT_{n-1} \int_{T_{n-1}}^\infty dT_n \Phi(T_1, T_2 - T_1, \dots) \delta(T - T_n),$$

where now the δ -function drops out by integrating over T_n :

$$(13) \quad \boxed{\varrho_n^*(E, 0) = (4\pi)^n \int_0^T dT_1 \int_{T_1}^T dT_2 \dots \int_{T_{n-2}}^T dT_{n-1} \Phi(T_1, T_2 - T_1, \dots, T_n - T_{n-1}).}$$

3. - The MC approach.

We consider the following process: We choose independently and at random $n - 1$ numbers between 0 and T in such a way that the probability density for the single numbers is constant over the whole interval; we then order them according to magnitude.

With N such ordered samples $(T_1^{(i)}, T_2^{(i)} \dots T_{n-1}^{(i)}); (i=1 \dots N)$ we have

$$(14) \quad \int_0^T dT_1 \int_{T_1}^T dT_2 \dots \int_{T_{n-2}}^T dT_{n-1} \Phi(T_1, T_2 - T_1, \dots) = \\ = \frac{T^{n-1}}{(n-1)!} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \Phi(T_1^{(i)}, T_2^{(i)} - T_1^{(i)}, \dots).$$

To prove the truth of (14), we note that the probability that the i -th ordered (*) sample fills just the cells $\Delta T_1, \Delta T_2, \dots, \Delta T_{n-1}$ is (see Fig. 1)

$$(n-1)! \frac{\Delta T_1}{T} \cdot \frac{\Delta T_1}{T} \dots \frac{\Delta T_{n-1}}{T} = \frac{(n-1)!}{T^{n-1}} \Delta T_1 \Delta T_2 \dots \Delta T_{n-1}$$

and N times this probability is the total number of ordered samples filling these cells — they all lead to the same value of φ . Hence (14) holds.

This is the principle — it remains to write the formulae in a more convenient form.

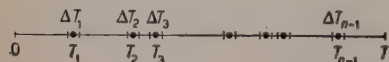


Fig. 1. — For the proof of (14).

It should be mentioned that one may hope for a fairly rapid convergence, since the sampling process has a tendency to draw preferably samples with

$T_1 \approx T_2 - T_1 \approx \dots \approx T - T_{n-1}$ because for large n the points $T_i^{(j)}$ cluster around the value $(i/n)T$. Fortunately, it happens that the integrand has a maximum for those values of the variables that nearly correspond to equipartition of kinetic energies. Thus the samples tend automatically to pick out the most important terms of the sum and to neglect the small ones. This is the main advantage of MC applied to this problem as compared to a simple multidimensional numerical integration. The problem of convergence will be treated in the following paper.

4. — The computing programme.

From (14) we have with (13)

$$(16) \quad \varrho_n^*(E, 0) = \frac{(4\pi)^n}{(n-1)!} T^{n-1} \cdot \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \Phi(T_1^{(i)}, T_2^{(i)} - T_1^{(i)}, \dots),$$

where $T_1^{(i)} \dots T_{n-1}^{(i)}$ is the i -th ordered sample. Going back to the variables ε_j , each ordered sample $T_1^{(i)} \dots T_{n-1}^{(i)}$ yields a sample $\varepsilon_1^{(i)} \dots \varepsilon_n^{(i)}$:

$$(17) \quad (\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \dots, \varepsilon_n^{(i)}) \equiv (T_1^{(i)} + m_1, T_2^{(i)} - T_1^{(i)} + m_2, \dots, T - T_{n-1}^{(i)} + m_n)$$

and

$$\Phi(T_1, T_2 - T_1, \dots, T - T_{n-1}) \equiv \varepsilon_1 \sqrt{\varepsilon_1^2 - m_1^2} \dots \varepsilon_n \sqrt{\varepsilon_n^2 - m_n^2} \cdot v_n(0, \varepsilon_1 \varepsilon_2 \dots \varepsilon_n).$$

(*) The ordering is responsible for the factor $(n-1)!$

For v_n we take the form (cf. A.9))

$$v_n(0, \varepsilon_1 \dots \varepsilon_n) = - \frac{1}{(n-3)! \pi \cdot 2^{n+1}} \cdot \frac{1}{\sqrt{\varepsilon_1^2 - m_1^2} \dots \sqrt{\varepsilon_n^2 - m_n^2}} \left[\sum_{\sigma_1 \dots \sigma_n} \sigma_1 \dots \sigma_n \left\{ \operatorname{sg} \left(\sum_i \sigma_i \sqrt{\varepsilon_i^2 - m_i^2} \right) \right\} \left(\sum_i \sigma_i \sqrt{\varepsilon_i^2 - m_i^2} \right)^{n-3} \right].$$

We have therefore

$$\Phi(T_1, T_2 - T_1, \dots, T - T_{n-1}) = \frac{1}{2^{n+1} \pi \cdot (n-3)!} \psi_n(\varepsilon_1 \dots \varepsilon_n)$$

with

$$(18) \quad \psi_n(\varepsilon_1 \dots \varepsilon_n) = - \varepsilon_1 \cdot \varepsilon_2 \dots \varepsilon_n \left[\sum_{\sigma_1 \dots \sigma_n} \sigma_1 \dots \sigma_n \left\{ \operatorname{sg} \left(\sum_i \sigma_i \sqrt{\varepsilon_i^2 - m_i^2} \right) \right\} \left(\sum_i \sigma_i \sqrt{\varepsilon_i^2 - m_i^2} \right)^{n-3} \right]$$

With this definition

$$(19) \quad \varrho_n^*(E, 0) = \frac{(E - M)^{n-1} \cdot (2\pi)^{n-1}}{(n-1)! (n-3)!} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \psi_n(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \dots, \varepsilon_n^{(i)}).$$

The samples $\varepsilon_1^{(i)}, \varepsilon_2^{(i)} \dots \varepsilon_n^{(i)}$ are constructed according to (17) and at the same time one calculates the corresponding set of square roots $\sqrt{\varepsilon_j^{(i)2} - m_j^2}$.

5. - The spectrum.

Another advantage of this MC method for solving the phase space problem is that by the very nature of the computation process one gets easily without further calculations the energy spectrum for each kind of particle considered, more accurately: that part of the spectrum which is determined by phase space considerations.

We consider here the (unnormalized) spectrum of the r -th particle, which because of the assumed individuality means a quite definite one. From (3) we see that in the C.M.S. ($P=0$) the quantity

$$(20) \quad dp_r \cdot 4\pi p_r^2 \cdot \varrho_{n-1}^*(E - \sqrt{p_r^2 + m_r^2}, p_r)$$

is proportional to the number of particles « r » with momentum between p_r and $p_r + dp_r$.

The function $\varrho_n^{(r)}(E, 0, \varepsilon_r)$, defined as follows:

$$(21) \quad \varrho_n^{(r)}(E, 0, \varepsilon_r) = 4\pi \varepsilon_r \sqrt{\varepsilon_r^2 - m_r^2} \cdot \varrho_{n-1}^*(E - \varepsilon_r, \sqrt{\varepsilon_r^2 - m_r^2})$$

allows then a similar interpretation.

Now from (9)

$$(22) \quad \varrho_{n-1}^*(E - \varepsilon_r, p_r) = (4\pi)^{n-1} \int_{m_1}^{\infty} d\varepsilon_1 \dots \int_{m_{r-1}}^{\infty} d\varepsilon_{r-1} \int_{m_{r+1}}^{\infty} d\varepsilon_{r+1} \dots \int_{m_n}^{\infty} d\varepsilon_n \cdot \\ \cdot u_1(\varepsilon_1) \dots u_{r-1}(\varepsilon_{r-1}) u_{r+1}(\varepsilon_{r+1}) \dots u_n(\varepsilon_n) \cdot v_{n-1}(p_r, \varepsilon_1, \dots, \varepsilon_{r-1}, \varepsilon_{r+1}, \dots, \varepsilon_n) \cdot \\ \cdot \delta(E - \varepsilon_r - \sum_{i \neq r} \varepsilon_i).$$

Furthermore with $\varepsilon_r = \sqrt{p_r^2 + m_r^2}$ from (A.7)

$$(23) \quad v_{n-1}(p_r, \varepsilon_1 \dots \varepsilon_{r-1}, \varepsilon_{r+1} \dots \varepsilon_n) = v_n(0, \varepsilon_1 \dots \varepsilon_n).$$

Since also $\delta(E - \varepsilon_r - \sum_{i \neq r} \varepsilon_i) = \delta(E - \sum_i \varepsilon_i)$, we see that the differential spectrum

of particle «*r*», namely $\varrho_n^{(r)}(E, 0, \varepsilon_r)$ is obtained from (9) if $d\varepsilon_r$ is taken outside the integrals, i.e., if the integration over ε_r is omitted. From this follows then the almost obvious way to calculate $\varrho_n^{(r)}(E, 0, \varepsilon_r)$.

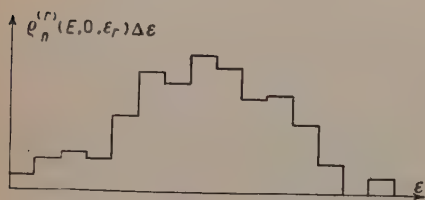


Fig. 2. — Typical form of a MC spectrum.

For each kind «*r*» of particle a set of *Z* storages is provided and the total energy *E* is divided into *Z* intervals $\Delta\varepsilon = E/Z$, each cell corresponding to one storage.

At the same time as the samples $\varepsilon_1^{(i)} \dots \varepsilon_n^{(i)}$ are prepared, one looks in each sample for the value of $\varepsilon_r^{(i)}$ and determines in which of the cells $\Delta\varepsilon$ it lies. Then one adds into the corresponding storage the number $\psi_n(\varepsilon_1^{(i)} \dots \varepsilon_n^{(i)})$ (see (19)).

After completion of this distribution of ψ 's (which takes place of course during the calculation of $\varrho^*(E, 0)$) into the cells, the whole distribution is multiplied by

$$\frac{1}{N} \cdot \frac{(E - M)^{n-1} (2\pi)^{n-1}}{(n-1)! (n-3)!}$$

and the result is $\varrho_n^{(r)}(E, 0, \varepsilon_r)$, the spectrum for the particle «*r*». Indeed: this distribution means nothing more than the omission of the integration over $d\varepsilon_r$ and represents the integrand as a function of ε_r .

The accuracy will of course be smaller than for $\varrho_n^*(E, 0)$ itself, because of the subdivision of the total statistical information, but one obtains without additional work at least a rough idea of the spectra.

In a following paper we shall give details of the actual performance of such a calculation including estimates of errors and a flow diagram for an electronic computer. A brief report concerning our experiences in practical calculations will be added.

APPENDIX

The random walk function.

Given a set of n vectors with fixed lengths $p_1 \dots p_n$ but random directions $\mathbf{e}_1 \dots \mathbf{e}_n$. What is the probability $w_n(\mathbf{P}, p_1 \dots p_n) d\mathbf{P}$, that the resultant $\sum_{i=1}^n p_i \mathbf{e}_i$ lies in the neighbourhood $d\mathbf{P}$ of \mathbf{P} ?

First we make two simplifying remarks:

1) Since each \mathbf{e}_i has a random direction, the same holds for the resultant. $w(\mathbf{P}, p_1 \dots p_n)$ can only depend on $|\mathbf{P}|$. We write therefore $w_n(P, p_1 \dots p_n)$.

2) The order in which the vectors are added is irrelevant, w_n must be symmetric in $p_1 \dots p_n$. Even more: the probability density for n vectors $\mathbf{p}_1 \dots \mathbf{p}_n$ to have a resultant \mathbf{P} must be the same as for the $n+1$ vectors $\mathbf{P}, \mathbf{p}_1 \dots \mathbf{p}_n$ to have the resultant zero. Hence

$$w_{n+1}(0, p_1 \dots p_n, P) = w_n(P, p_1 \dots p_n)$$

and since the left hand side is symmetric in p , this must be true also. P for the r.h.s.

We now derive Eq. (5).

The required probability is obviously proportional to the number of possible realizations, *viz.*

$$(A.1) \quad w_n(P, p_1 \dots p_n) = K_n \int \delta(\mathbf{P} - \sum p_i \mathbf{e}_i) d\mathbf{e}_1 \dots d\mathbf{e}_n,$$

where we have imposed the condition by means of a δ -function and integrate thus over all possible ways of choosing directions, which give the wanted length of the resultant. K_n is a normalizing factor to be determined later.

Introducing a Fourier representation for the δ -function

$$(A.2) \quad \delta(\mathbf{P} - \sum_i p_i \mathbf{e}_i) = \left(\frac{1}{2\pi}\right)^3 \int \exp[i\boldsymbol{\lambda}(\mathbf{P} - \sum p_i \mathbf{e}_i)] d\boldsymbol{\lambda},$$

we find

$$w_n(P, p_1 \dots p_n) = \frac{K_n}{(2\pi)^3} \int d\boldsymbol{\lambda} d\mathbf{e}_1 \dots d\mathbf{e}_n \exp[i\boldsymbol{\lambda}(\mathbf{P} - \sum p_i \mathbf{e}_i)]$$

or

$$(A.3) \quad w_n(P, p_1 \dots p_n) = \frac{K_n}{(2\pi)^3} \int d\boldsymbol{\lambda} \exp[i\boldsymbol{\lambda}\mathbf{P}] \cdot \prod_{j=1}^n \left[\int d\mathbf{e} \exp[-ip_j \boldsymbol{\lambda}\mathbf{e}] \right].$$

With polar co-ordinates $d\mathbf{e} = 2\pi \sin \theta d\theta$ one has

$$\int d\mathbf{e} \exp[-ip_j \boldsymbol{\lambda}\mathbf{e}] = \frac{4\pi \sin p_j \lambda}{p_j \lambda}.$$

Since the product does no longer depend on the direction of λ we may integrate $\exp[i\lambda\mathbf{P}]$ also over all directions of λ with the same result and find

$$(A.4) \quad w_n(P, p_1 \dots p_n) = \frac{K_n}{(2\pi)^3} (4\pi)^{n+1} \int_0^\infty \lambda^2 d\lambda \frac{\sin P\lambda}{P\lambda} \cdot \frac{\sin p_1\lambda}{p_1\lambda} \dots \frac{\sin p_n\lambda}{p_n\lambda}.$$

The normalization requires

$$(A.5) \quad \lim_{P_0 \rightarrow \infty} \int_0^{P_0} w_n(P, p_1 \dots p_n) \cdot 4\pi P^2 dP = 1.$$

(It is necessary to consider this limit, because we have reversed the order of integration and thereby obtained a singular function.) We obtain

$$4\pi \int_0^{P_0} w_n(P, p_1 \dots p_n) P^2 dP = \frac{K_n (4\pi)^{n+2}}{(2\pi)^3} \int_0^\infty \lambda d\lambda \frac{\sin p_1\lambda}{p_1\lambda} \dots \frac{\sin p_n\lambda}{p_n\lambda} \left[\frac{\sin P_0\lambda}{\lambda^2} - \frac{P_0}{\lambda} \cos P_0\lambda \right].$$

Now the second term in the bracket makes the integral vanish for $P_0 \rightarrow \infty$ whereas the first one gives

$$4\pi \int_0^{P_0} w_n(P, p_1 \dots p_n) P^2 dP = \frac{K_n (4\pi)^{n+2}}{(2\pi)^3} \cdot \frac{\pi}{2} \cdot \frac{1}{\pi} \int_{-\infty}^{+\infty} d\lambda \frac{\sin \lambda P_0}{\lambda} \frac{\sin \lambda p_1}{\lambda p_1} \dots \frac{\sin \lambda p_n}{\lambda p_n}.$$

Since

$$\lim_{P_0 \rightarrow \infty} \frac{1}{\pi} \frac{\sin \lambda P_0}{\lambda} = \delta(\lambda),$$

we find

$$4\pi \int_0^\infty w_n(P, p_1 \dots p_n) P^2 dP = (4\pi)^n \cdot K_n = 1.$$

Hence the normalized random walk function is

$$(A.6) \quad w_n(P, p_1 \dots p_n) = \left(\frac{1}{4\pi} \right)^n \int \delta(\mathbf{P} - \sum p_i \mathbf{e}_i) d\mathbf{e}_1 \dots d\mathbf{e}_n = \\ = \frac{1}{2\pi^3} \int_0^\infty \lambda^2 d\lambda \frac{\sin P\lambda}{P\lambda} \frac{\sin p_1\lambda}{p_1\lambda} \dots \frac{\sin p_n\lambda}{p_n\lambda}.$$

We observe at once the complete symmetry not only with respect to $p_1 \dots p_n$ but also with respect to $P, p_1 \dots p_n$, which all may be arbitrarily permuted. A simple consequence is the important equation

$$(A.7) \quad w_n(0, p_1 \dots p_n) = w_{n-1}(p_j, p_1 \dots p_{j-1}, p_{j+1} \dots p_n); \quad j = 1 \dots n.$$

which we found already at the beginning from general arguments.

We now evaluate (A.6) for $P=0$ (which, because of (A.7), is no restriction). Written with exponentials and the λ -integration going from $-\infty$ to ∞ , (A.6) gives

$$w_n(0, p_1 \dots p_n) = \frac{1}{4\pi^2} \cdot \frac{1}{p_1 \dots p_n} \cdot \left(\frac{1}{2i} \right)^n \int_{-\infty - i\tau}^{\infty - i\tau} \frac{d\lambda}{\lambda^{n-2}} (\exp[ip_1 \lambda] - \exp[-ip_1 \lambda]) \dots (\exp[ip_n \lambda] - \exp[-ip_n \lambda]) .$$

Since the integrand does in fact contain no singularity, we could shift the path by $-i\tau$ into the lower half plane. Evaluating the product and closing then the path in the upper or lower half plane according to where the integrand vanishes, we find for the integral

$$\int_{-\infty - i\tau}^{\infty - i\tau} \frac{d\lambda}{\lambda^{n-2}} \sum_{\sigma} \sigma_1 \dots \sigma_n \exp[i\lambda \sum_i \sigma_i p_i] = \frac{2\pi i}{(n-3)!} \frac{d^{n-3}}{d\lambda^{n-3}} \left\{ \sum_{\sigma} \sigma_1 \dots \sigma_n \exp[i\lambda \sum_i \sigma_i p_i] \right\}_{\substack{\sum \sigma_i p_i \geq 0 \\ \lambda=0}} .$$

Here every σ takes the two values $+1$ and -1 and the sum is over all 2^n sign combinations, except those for which $\sum \sigma_i p_i < 0$. (Because for these the path is closed in the lower half plane with residual zero). The differentiation gives finally

$$(A.8) \quad w_n(0, p_1 \dots p_n) = - \frac{1}{(n-3)!} \frac{1}{2^{n+1}\pi} \cdot \frac{1}{p_1 \dots p_n} \left[\sum_{\sigma} \sigma_1 \dots \sigma_n (\sum_i \sigma_i p_i)^{n-3} \right]_{\sum \sigma_i p_i \geq 0} .$$

For practical calculation of the sum over σ we remark that to every sign combination there exists the inverse one and one of them gives $\sum_i \sigma_i p_i \geq 0$.

We need therefore sum only over one half of all sign combinations provided that we reverse all signs if $\sum_i \sigma_i p_i < 0$. We may therefore fix $\sigma_1 \equiv +1$ and vary all the rest. This gives just one half of the possible sign combinations $\sigma_1 \dots \sigma_n$ and no two of them are the inverse of each other. Thus the whole class $[\sigma_1 = +1]$ is inverse to the whole class $[\sigma_1 = -1]$ and we shall now sum only over the former one. If for a certain combination $\sigma_1 \dots \sigma_n$ ($\sigma_1 = +1$) one finds $\sum \sigma_i p_i < 0$, one has to change all signs $\sigma_i \rightarrow \sigma'_i = -\sigma_i$ and to add

$$\sigma'_1 \sigma'_2 \dots \sigma'_n (\sum \sigma'_i p_i)^{n-3} = (-)^n \sigma_1 \dots \sigma_n \cdot (-)^{n-3} (\sum \sigma_i p_i)^{n-3} = -\sigma_1 \dots \sigma_n (\sum \sigma_i p_i)^{n-3} .$$

That means: If $\sum \sigma_i p_i < 0$, we have to subtract $\sigma_1 \dots \sigma_n (\sum \sigma_i p_i)^{n-3}$ instead of adding it. This is expressed simply by writing

$$(A.9) \quad w_n(0, p_1 \dots p_n) = - \frac{1}{(n-3)!} \frac{1}{2^{n+1}\pi} \cdot \frac{1}{p_1 \dots p_n} \left[\sum_{\substack{\sigma_2 \dots \sigma_n \\ \sigma_1 = +1}} \sigma_1 \dots \sigma_n \cdot \text{sg}(\sum \sigma_i p_i) (\sum \sigma_i p_i)^{n-3} \right] ,$$

where

$$\text{sg } x = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x \leq 0 \end{cases}$$

is the sign function (*).

An evident consequence of the meaning of the random walk function is that it must vanish identically as soon as any p_j is larger than the sum of all the others:

$$(A.10) \quad w_n(0, p_1 \dots p_n) \equiv 0 \quad \text{if} \quad 2p_j - \sum_{i=1}^n p_i \quad (\text{for any } j=1 \dots n).$$

As this is not obvious in the present form (A.9) of the function, we prove it explicitly:

Because of the symmetry one may take $j=1$. The square bracket of (A.9) must vanish.

i) We observe that if $2p_1 \geq \sum p_i$, one has

$$\text{sg} \left(\sum \sigma_i p_i \right) \equiv +1.$$

ii) We expand the power in writing (\sum' means $i \neq 1$)

$$\begin{aligned} \sum_{\sigma} \sigma_2 \dots \sigma_n (p_1 + \sum' \sigma_i p_i)^{n-3} &= \sum_{\sigma} \sigma_2 \dots \sigma_n \sum_{\sum i_k = n-3} [p_1^{i_1} (\sigma_2 p_2)^{i_2} \dots (\sigma_n p_n)^{i_n}] = \\ &= \sum_{\sum i_k = n-3} p_1^{i_1} p_2^{i_2} \dots p_n^{i_n} \sum_{\sigma} \sigma_2^{i_2+1} \sigma_3^{i_3+1} \dots \sigma_n^{i_n+1}. \end{aligned}$$

iii) From $\sum_{k=1}^n i_k = n-3$ follows that not all $i_k \neq 0$. Let $i_j = 0$. Then one has (\sum'' means: $\sigma_j^{i_j+1}$ omitted)

$$\sum_{\sigma} = \sum_{\sigma_j = \pm 1} \sigma_j \left(\sum'' \sigma_i^{i_i+1} \dots \sigma_n^{i_n+1} \right) = \sum'' - \sum'' = 0. \quad q.e.d.$$

The evaluation of (A.9) seems very easy on an electronic computer, if one

(*) Ambiguities arise if accidentally a certain $\sum \sigma_i p_i = 0$. In fact this gives a difficulty only for $n=3$, since in that case $w(0, p_1 p_2 p_3)$ is a discontinuous function because of the triangle inequality. As a pure convention we put $\text{sg}(0) = -1$. For $n > 3$ one has in such a case $(\sum \sigma_i p_i)^{n-3} = 0$, which settles the question. In the programme of an electronic computer, however, this does not help the machine to know what to do with the $\text{sg}(0)$, so we shall adopt the same definition for all n .

proceeds in the following way:

a) One calculates $\sum_{i=1}^n p_i$, which corresponds to

$$\sigma_1 = \sigma_2 = \dots = \sigma_n = 1.$$

b) One varies the signs in such a way that in the order $\sigma_2 \sigma_3 \dots \sigma_n$ each σ keeps its sign as long as possible (see example):

$\sigma_1 \cdot \sigma_2 \dots \sigma_n$	σ_2	σ_3	σ_4	σ_5	Corresponds to
+	+	+	+	+	$p_1 + p_2 + p_3 + p_4 + p_5$
-	+	+	+	-	$-2p_5$
+	+	+	-	-	$-2p_4$
-	+	+	-	+	$+2p_5$
+	+	-	-	+	$-2p_3$
-	+	-	-	-	$-2p_5$
+	+	-	+	-	$+2p_4$
-	+	-	+	+	$+2p_4$
+	-	-	+	+	$-2p_2$
-	-	-	+	-	$-2p_5$
+	-	-	-	-	$-2p_4$
-	-	-	-	+	$+2p_5$
+	-	+	-	+	$+2p_3$
-	-	+	-	-	$-2p_5$
+	-	+	+	-	$+2p_4$
-	-	+	+	+	$+2p_3$

Change against foregoing set $\sum \sigma_i p_i$

The example shows that from combination to combination there is always only one addition to do and the sign of $\sigma_1 \cdot \sigma_2 \dots \sigma_n$ alternates systematically. Besides the additions, the machine has to do the $(n-3)$ rd power and to ob-

serve the sign of $\sum \sigma_i p_i$. The total number of additions goes therefore roughly with 2^{n-1} and after each addition (*i.e.* for each $\sum \sigma_i p_i$) the power is to be calculated, that is 2^{n-1} times a $(n-3)$ rd power. We are unlikely to have to calculate cases where n is bigger than 12, the normal situation will have n running from 3 to 10 (only mesons produced) and from 3 to 5 or 6 (heavier particles produced). For $n=12$ one has thus ~ 2050 additions and 2050 times a 9-th power. Cases with $n=7$ may be most interesting. This gives ~ 150 additions and 150 times a 4-th power.

Since this number of operations has to be carried out for each sample of $p_1 \dots p_n$, the calculation of $w_n(0, p_1 \dots p_n)$ contributes a great deal to the total computing time.

A Monte-Carlo Method to Calculate Multiple Phase Space Integrals - II.

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(ricevuto il 18 Agosto 1958)

CONTENTS. — 1. Error estimates. — 2. Flow diagram. — 3. Practical experiences. — 4. Estimates of computing time. — 5. Comparison with the usual approximation.

1. — Error estimates.

In a foregoing paper ⁽¹⁾, which will be quoted henceforward as (I), we presented a Monte-Carlo (MC) approach to the problem of calculating multiple phase space integrals. We shall use the same notation as in (I).

The proposed calculation can be done only on automatic computers and of course only with a limited accuracy. As a MC calculation is a statistical process, one must know how accurate its result actually—or better: probably—is. Automatic computers allow one to include a check of accuracy within the programme, so that the machine stops or goes over to the next case as soon as the prescribed accuracy is reached.

We therefore begin with an estimate of the error. This error estimate applies to a single calculation of a particular $\varrho_n^*(E, 0)$. Actually, however, one will calculate this function for $n = 3, 4, 5 \dots$ and also perhaps for different energies. Since we know that it is a smooth function with one single maximum if we vary either n or E (keeping the other fixed), we may get a better

⁽¹⁾ *A Monte-Carlo method to calculate multiple phase space integrals* (I). Equations of that paper are quoted as *e.g.* (I, 15), whereas equations of this paper are referred to as *e.g.* (12), etc.

statistical accuracy for a whole table, than we would have for a single case. If, *e.g.* for fixed E , one calculates a table for $n=3, 4, \dots, n'$, then the accuracy of a number from such a table will be better by a factor of roughly $1/\sqrt{n'-3}$ provided the values in the table are smoothed out by using any method of balancing the fluctuations.

For the estimation of the individual error we proceed as follows:

First a certain number N_0 is fixed ($N_0 \gg 1$) which is the basic number of samples or the basic number of contributions to the sum (I, 19). This number should be chosen so small—say 10 or 50 or 100—that one does not expect that N_0 terms make already a good approximation to ϱ^* . The process has then to be repeated several times and its convergence watched. Each repetition will add again N_0 contributions. Thus, if the desired accuracy is reached with λ sets of N_0 contributions, the number of samples is $N = \lambda N_0$. We introduce the following abbreviations:

- | | | |
|-----|--|--|
| (1) | $\varrho_\lambda = \varrho_n^{*(\lambda)}(E, 0)$ | <i>i.e.</i> the MC approximation to $\varrho_n^*(E, 0)$ ⁽²⁾ with $N = \lambda N_0$ contributing samples (see (I, 19)). We want this approximation to fulfil some condition of accuracy to be specified later. |
| (2) | $S_\mu = \sum_{i=(\mu-1)N_0+1}^{\mu N_0} \psi(\varepsilon_1^{(i)} \dots \varepsilon_n^{(i)})$ | <i>i.e.</i> the partial sum of the ψ 's from the μ -th set of N_0 samples |
| (3) | $\Sigma_\lambda = \sum_{\mu=1}^{\lambda} S_\mu = \sum_{i=1}^{\lambda N_0} \psi(\varepsilon_1^{(i)} \dots \varepsilon_n^{(i)})$ | <i>i.e.</i> the total sum of ψ 's from the $N = \lambda N_0$ samples. |

With these definitions it follows from (I, 19) that,

$$(4) \quad \varrho_\lambda = \frac{A}{\lambda} \Sigma_\lambda; \quad A = \frac{(E-M)^{n-1} (2\pi)^{n-1}}{(n-1)! (n-3)! N_0}; \quad \varrho_n^*(E, 0) = \lim_{\lambda \rightarrow \infty} \varrho_\lambda \equiv \varrho.$$

Consider first λ as a fixed number and suppose that ϱ_λ has been calculated. To get an estimate of the accuracy we can use only the accumulated data from the λN_0 samples and we therefore deal with *a posteriori* estimates only.

We can say what would be the mean error (more accurately its r.m.s. value ⁽³⁾) if we would calculate ϱ_λ many times independently. In fact, we calculate it only once and may have found a value which differs much more from the exact value than just by one mean error. There is no way out of this si-

⁽²⁾ We omit throughout the star and write simply ϱ_λ and ϱ for convenience. The quantities, however, correspond to ϱ^* as defined in (I).

⁽³⁾ r.m.s. means: root mean square.

tuation, because it is a logical consequence of the MC method, and common to all sampling procedures. Even if we would really calculate many times and take the mean value, then this mean value would simply be another single (though more accurate) calculation of $\varrho_{\lambda'}$ where now λ' is many times larger than λ was. We can then carry through the same reasoning for λ' , which we gave above for λ , and we come to precisely the same result. Only the mean error will be smaller.

Let us consider a typical situation. If ϱ_{λ} has been calculated, then this is the mean value of λ contributions S_{μ} (all independent of each other), and each one of these comes from N_0 samples. We may here forget about the latter fact, because N_0 influences only the magnitude of the fluctuations of the S_{μ} . We shall consider therefore the S_{μ} as random variables. We draw a histogram, which shows this situation:

$$\frac{\varrho_{\lambda}}{A} = S^{(\lambda)} = \frac{1}{\lambda} \sum_{\mu=1}^{\lambda} S_{\mu}$$

is the mean value and it is supposed that

$$\lim_{\lambda \rightarrow \infty} \frac{\varrho_{\lambda}}{A} = \frac{\varrho}{A}.$$

Actually we do not know this limit and the only quantities which are at our disposal are those which we can extract from the above picture.

For the following we need two different mean values:

i) The mean value over the λ contributions which have so far been calculated. This mean value can be inferred numerically from the actual data; it will be denoted by a bar,

ii) We may imagine the above actual case to be one of a large ensemble, *i.e.* we may imagine ϱ_{λ} to be calculated L times in the same way (statistically independent) and all the L histograms drawn. They will yield L values $\varrho_{\lambda}^{(l)}$ ($l=1 \dots L$). The second kind of mean value is then over the L -ensemble. Here we take the limit $L \rightarrow \infty$ and write $\langle \rangle$.

The contributions S_{μ} have from now on a second label l such that $S_{\mu}^{(l)}$ is the μ -th contribution to the l -th histogram. Let F be any function of S , then i) and ii) mean

$$(5) \quad \left\{ \begin{array}{ll} \text{i)} & \overline{F^{(l)}}^{(\lambda)} = \frac{1}{\lambda} \sum_{\mu=1}^{\lambda} F(S_{\mu}^{(l)}); \quad \lambda \text{ always finite} \\ \text{ii)} & \langle F \rangle = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{l=1}^L F(S_{\mu}^{(l)}). \end{array} \right.$$

Whereas $\overline{F^{(l)}}^{(\lambda)}$ still depends on l , $\langle F \rangle$ is clearly independent of μ . Of course

$$\lim_{\lambda \rightarrow \infty} \overline{F^{(l)}}^{(\lambda)} = \overline{\langle F \rangle}^{(\lambda)} = \langle F \rangle,$$

but we are just interested in the deviations for finite λ .

We proceed now to estimate the error of q_λ as a function of λ . With $\lim_{\lambda \rightarrow \infty} q_\lambda = \langle q_\lambda \rangle = q = A \langle S \rangle$ we have

$$(q_\lambda - q)^2 = \frac{A^2}{\lambda^2} \left(\sum_{\mu=1}^{\lambda} S_\mu - q/A \right)^2 = \frac{A^2}{\lambda^2} \sum_{\nu, \mu}^{\lambda} \Delta S_\nu \Delta S_\mu; \quad \Delta S_\mu = S_\mu - \langle S \rangle.$$

Since all contributions are statistically independent, we find with (5)

$$\langle (q_\lambda - q)^2 \rangle = \frac{A^2}{\lambda^2} \sum_{\mu=1}^{\lambda} \langle (\Delta S_\mu)^2 \rangle = \frac{A^2}{\lambda^2} \sum_{\mu=1}^{\lambda} \langle \Delta S^2 \rangle = \frac{A^2}{\lambda} \langle \Delta S^2 \rangle.$$

We now define $(\delta q)_\lambda \equiv \sqrt{\langle (q_\lambda - q)^2 \rangle}$ and have

$$(6) \quad \frac{(\delta q)_\lambda}{q} \equiv \frac{\sqrt{\langle (q_\lambda - q)^2 \rangle}}{q} = \frac{1}{\sqrt{\lambda}} \frac{\sqrt{\langle \Delta S^2 \rangle}}{\langle S \rangle}.$$

In the following discussion we shall assume that the distribution of S_μ around S is more or less a Gaussian distribution. We use this assumption only for estimates of orders of magnitude. If N_0 is large enough, the fluctuations of the S_μ will be small and in the neighbourhood of $\langle S \rangle$ the Gauss distribution will be a good approximation. This has the advantage that we can calculate some quantities explicitly. Though the results do not hold strictly, they represent at least good estimates.

If $\langle \Delta S^2 \rangle$ (and hence $\delta q = \sqrt{\langle \Delta q^2 \rangle}$) were known, we could state confidence limits for q , starting from the calculated q_λ . Assuming a nearly Gaussian distribution for S_μ it is well-known that

$$\begin{array}{lll} \text{the probability that} & |q - q_\lambda| \leq \delta q, & \text{is } \approx 68\% \\ \text{»} & \text{»} & \text{» } |q - q_\lambda| \leq 2 \cdot \delta q, \text{ is } \approx 95\% \\ \text{»} & \text{»} & \text{» } |q - q_\lambda| \leq 3 \cdot \delta q, \text{ is } \approx 99.95\%. \end{array}$$

Then, in order to obtain a given probability that the relative error is smaller than a given (small) number, we only have to choose the corresponding λ large enough. Unfortunately neither $\langle \Delta S^2 \rangle$ nor $\langle S \rangle$ are actually known. We may, however, without great error replace in (6) $\langle S \rangle$ by $\overline{S^{(2)}}$, because $\delta q/q$

will be changed only by an amount of the second order. So we redefine

$$(7) \quad \frac{(\delta \varrho)_\lambda}{\varrho} = \frac{1}{\sqrt{\lambda}} \frac{\sqrt{\langle \Delta S^2 \rangle}}{\bar{S}^{(\lambda)}}.$$

There are now many ways to estimate $\langle \Delta S^2 \rangle$ from the sample of λ elements S_μ , *e.g.* using the largest occurring deviation or the r.m.s. deviation from the mean value. These methods, of which Student's test is the most adequate, have several practical drawbacks, the main one being that after each addition of a new set of samples the mean value changes and all deviations have to be calculated anew.

These disadvantages may be avoided in the following way: We use the quantity $\bar{\delta S}^{(\lambda)}$ which we define as follows:

$$(8) \quad \bar{\delta S}^{(\lambda)} = \frac{1}{\lambda} \sum_{\mu=2}^{\lambda} \left(\frac{1}{\mu} \sum_{\tau=1}^{\mu} S_\tau \right) - S_\mu \quad \left| \quad \frac{1}{\lambda} \sum_{\mu=2}^{\lambda} | \bar{S}^{(\mu)} - S_\mu | \right|.$$

That means: After each step of the calculation we take the mean value $\bar{S}^{(\mu)}$ and find the absolute value of the deviation of the last contribution S_μ from this mean value. The average of all these absolute values is our $\bar{\delta S}^{(\lambda)}$. The advantage is that the mean values $\bar{S}^{(\mu)}$ are calculated anyway, because they yield the ϱ_μ , and S_μ is just calculated as the last contribution. The absolute value of only this one difference is taken, added to the corresponding quantities of the foregoing steps and the mean is taken; neither are new differences calculated, nor are square roots involved. Since $\bar{S}^{(\mu)}$ tends to $\langle S \rangle$ as $\mu \rightarrow \infty$, it is easy to show that

$$(9) \quad \lim_{\lambda \rightarrow \infty} \bar{\delta S}^{(\lambda)} = \langle |\Delta S| \rangle.$$

But, since we do not intend to use large λ , we must relate the new quantity to $\langle |\Delta S| \rangle$ or $\langle \Delta S^2 \rangle$ also for finite λ .

For this purpose we use the assumption of a Gauss distribution, since this allows explicit calculations. We shall calculate the expectation value of $\bar{\delta S}^{(\lambda)}$ and express it by $\langle \Delta S^2 \rangle$ under the assumption that the distribution of S_μ about $\langle S \rangle$ follows a Gauss law

$$(10) \quad \Delta S = S - \langle S \rangle$$

and

$$(11) \quad w(S) = \frac{1}{\sqrt{2\pi\langle \Delta S^2 \rangle}} \exp \left[-\frac{\Delta S^2}{2\langle \Delta S^2 \rangle} \right]; \quad \int_{-\infty}^{+\infty} w(S) dS = 1.$$

Then the expectation value (5 ii) is defined as

$$(12) \quad \langle F \rangle = \int_{-\infty}^{+\infty} F(S) w(S) dS.$$

We need for the following a simple consequence of (11): As is well known, the mean value of a sum of random variables follows again a Gauss distribution, if the random variables themselves follow such a distribution. The r.m.s. deviation of the mean value of $\bar{S}^{(n)}$ from $\langle S \rangle$ is then, as shown in the derivation of (6)

$$(13) \quad \langle (\bar{S}^{(n)} - \langle S \rangle)^2 \rangle \equiv \langle \Delta_n S^2 \rangle = \frac{1}{n} \langle \Delta S^2 \rangle.$$

The probability that $\bar{S}^{(n)}$ lies between x and $x+dx$, is therefore according to (11) and (13) given by

$$(14) \quad W_n(x) dx = \frac{dx}{\sqrt{2\pi \cdot (1/n) \langle \Delta S^2 \rangle}} \exp \left[-\frac{(x - \langle S \rangle)^2}{2(1/n) \langle \Delta S^2 \rangle} \right].$$

We determine now the probability distribution of the difference

$$(15) \quad \sigma_\mu \equiv \bar{S}^{(\mu)} - S_\mu$$

in (8). We must take care of the fact, that $\bar{S}^{(\mu)}$ is not independent of S_μ and write, therefore

$$\sigma_\mu = \bar{S}^{(\mu)} - S_\mu = \frac{1}{\mu} \left(\sum_{\tau=1}^{\mu-1} S_\tau + S_\mu \right) - S_\mu = \frac{\mu-1}{\mu} (\bar{S}^{(\mu-1)} - S_\mu),$$

where now $\bar{S}^{(\mu-1)}$ is independent of S_μ . We put $\bar{S}^{(\mu-1)} - S_\mu = y_\mu$. The probability that $y \leq y_\mu \leq y + dy$, is then obviously

$$v_\mu(y) dy = dy \int_{-\infty}^{+\infty} W_{\mu-1}(y + S_\mu) w(S_\mu) dS_\mu$$

and with (14) and (11):

$$(16) \quad v_\mu(y) = \frac{1}{\sqrt{(\mu/(\mu-1)) 2\pi \langle \Delta S^2 \rangle}} \exp \left[-\frac{y^2}{(\mu/(\mu-1)) 2 \langle \Delta S^2 \rangle} \right].$$

The distribution of σ_μ is then according to the general formula

$$V\{f(x)\} df(x) = v(x) dx$$

given by

$$(17) \quad V_{\mu}(\sigma) = \frac{1}{\sqrt{((\mu-1)/\mu) 2\pi\langle\Delta S^2\rangle}} \exp\left[-\frac{\sigma^2}{((\mu-1)/\mu) 2\langle\Delta S^2\rangle}\right].$$

This is again a Gauss distribution with, as a glance on (11) shows:

$$(18) \quad \langle\sigma_{\mu}^2\rangle = \frac{\mu-1}{\mu} \langle\Delta S^2\rangle.$$

A simple calculation shows now that for a Gauss distribution about zero

$$(19) \quad \langle|x|\rangle = 2 \int_0^{\infty} x w(x) dx = \sqrt{\frac{2}{\pi}} \sqrt{\langle x^2 \rangle}.$$

This gives

$$\langle|\bar{S}^{(\mu)} - S_{\mu}|\rangle = \sqrt{\frac{2}{\pi}} \sqrt{\frac{\mu-1}{\mu}} \sqrt{\langle\Delta S^2\rangle},$$

which we insert now in the expectation value of (8), namely

$$\langle\bar{\delta S}^{(\lambda)}\rangle = \frac{1}{\lambda} \sum_{\mu=2}^{\lambda} \langle|\bar{S}^{(\mu)} - S_{\mu}|\rangle,$$

with the result

$$(20) \quad \langle\bar{\delta S}^{(\lambda)}\rangle = \sqrt{\frac{2}{\pi}} \sqrt{\langle\Delta S^2\rangle} \frac{1}{\lambda} \sum_{\mu=2}^{\lambda} \sqrt{\frac{\mu-1}{\mu}}$$

or

$$(21) \quad \sqrt{\langle\Delta S^2\rangle} = \sqrt{\frac{\pi}{2}} \frac{\langle\bar{\delta S}^{(\lambda)}\rangle}{\frac{1}{\lambda} \sum_{\mu=2}^{\lambda} \sqrt{\frac{\mu-1}{\mu}}} = \sqrt{\frac{\pi}{2}} \frac{\langle\sum_{\mu=2}^{\lambda} |\bar{S}^{(\mu)} - S_{\mu}|\rangle}{\sum_{\mu=2}^{\lambda} \sqrt{\frac{\mu-1}{\mu}}}.$$

This can now be introduced into (7). Of course, we find ourselves here before the same difficulty as with (7). What we actually know is not $\langle\bar{\delta S}^{(\lambda)}\rangle$, but only that particular $\bar{\delta S}^{(\lambda)}$ which we can calculate using the actually computed S_{μ} . As pointed out at the beginning, this is a logical consequence of the fact that we do not know the distribution, but only a finite number of drawn samples. We could now continue the argument and proceed to calculate the distribution of $\bar{\delta S}^{(\lambda)}$ around its expectation value $\langle\bar{\delta S}^{(\lambda)}\rangle$. We would then end up with essentially the same situation.

In fact the situation is not so bad; if we now replace

$$\langle \overline{\delta S^{(\lambda)}} \rangle$$

by the calculated value

$$\overline{\delta S^{(\lambda)}}$$

then for not too small λ the actual $\overline{\delta S^{(\lambda)}}$ lies near its expectation value. The error in the error estimate, which we thereby introduce, is therefore only of higher order. Even if we would estimate the error wrongly by a factor 2 or so (which is very improbable), this would not cause any serious trouble. We shall therefore adopt the following convention for the estimate of the error:

$$(22) \quad \frac{(\delta \varrho)_\lambda}{\varrho} = \sqrt{\frac{\pi}{2\lambda}} \cdot \frac{1}{\overline{\delta S^{(\lambda)}}} \cdot \frac{\sum_{\mu=2}^{\lambda} |\overline{\delta S^{(\mu)}} - S_\mu|}{\sum_{\mu=2}^{\lambda} \sqrt{\frac{\mu-1}{\mu}}} \equiv z_\lambda \leq z_0; \quad \lambda \geq \lambda_0 \gg 1.$$

Here z_0 is a given small number fixing the accuracy. $\lambda_0 \gg 1$ means λ_0 greater than at least 5. This is necessary, since otherwise the mean value over the actual deviations does not mean very much. Note that for $\lambda = 1$ the expressions would not be defined since numerator and denominator become zero. The sum in the denominator,

$$\sum_{\mu=2}^{\lambda} \sqrt{\frac{\mu-1}{\mu}},$$

can be calculated for $\lambda = 5, 6, \dots, 20$, or so, and be stored in the computer. Then no square roots are necessary during the programme and the check of the accuracy by means of (22) becomes extremely simple in an automatic calculation.

If, depending on the construction of the computer, it is simpler to take a square than to take the absolute value of a given number, then one can use instead of (22) the following formula, which is derived in essentially the same way:

$$(23) \quad \left(\frac{\delta \varrho}{\varrho} \right)_\lambda = \frac{1}{\lambda} \left(\frac{1}{\overline{\delta S^{(\lambda)}}} \right)^2 \frac{\sum_{\mu=2}^{\lambda} (\overline{\delta S^{(\mu)}} - S_\mu)^2}{\sum_{\mu=2}^{\lambda} \frac{\mu-1}{\mu}} = z_\lambda^2 \leq z_0^2; \quad \lambda \geq \lambda_0 \gg 1,$$

with the same z_0 as above. The factor $\sqrt{\pi/2}$ and the roots in the denominator have disappeared. The sum in the denominator can here be calculated for

each check by the recurrence formula, since the sum for $\lambda - 1$ is already there from the foregoing check.

It should be noted again that if

$$\frac{(\delta \varrho)_\lambda}{\varrho} = z_\lambda,$$

this means that the probability that the actual error is $\leq z_\lambda$ is $\approx 68\%$ and so on (see discussion following eq. (6)). Even that is only true for large λ , since we replaced $\langle \delta S^{(\lambda)} \rangle$ by the calculated $\delta \bar{S}^{(\lambda)}$. To be sure, one can always check a few results by Student's test and make z_0 smaller if necessary.

In the following flow diagram (22) and (23) are considered separately.

2. - Flow diagram.

The following flow diagram has been developed in collaboration with members of the « Institut für praktische Mathematik » Technische Hochschule, Darmstadt, Germany.

It may therefore show some features which are due to the particular properties of the IBM 650 computer that was used for the calculation. Still, it is felt that this flow diagram will remain essentially the same for any type of computer. Therefore it was considered useful to reproduce it here, since it involves some little tricks.

A few comments:

a) Input data

T , N_0 , n , A , $\Delta \varepsilon$ are defined in (I).

λ_0 and z_0 are defined here by eqs. (22) and (23).

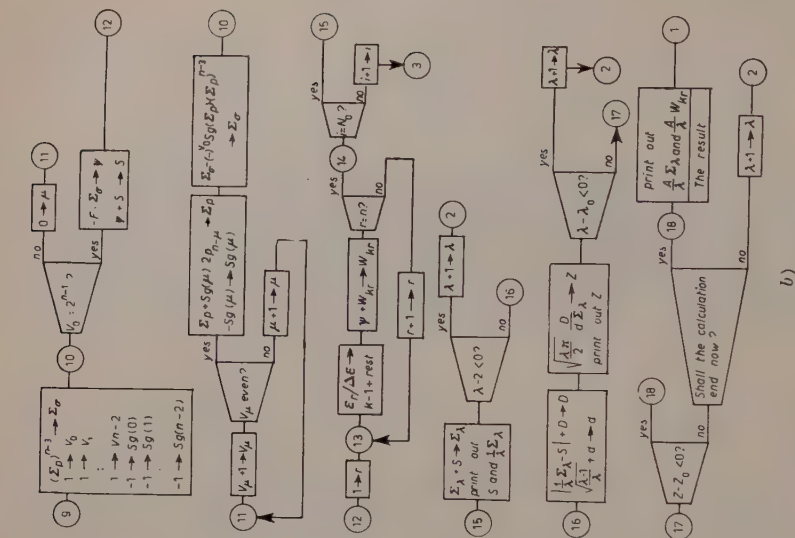
$m_1 \dots m_n$ are the masses of the particles.

The most convenient values for λ_0 and N_0 and reasonable values for z_0 depend on the total energy and on the number and masses of the particles. λ_0 should at least be of the order of 5. The smaller we choose N_0 , the larger λ becomes and the better one can observe the convergence: see discussion under h).

b) Instructions before (2)

W_{kr} gives the spectrum.

$(A/\lambda)W_{kr}$ will be the $\varrho_n^{(r)}(E, 0, \varepsilon_r)\Delta \varepsilon$ (of (I), Fig. 2) in the k -th cell. In this programme each particle gets its own spectrum, whether there are identical particles or not. If there are equal particles, one takes in the end the mean value of their spectra thus gaining statistical accuracy. It is convenient to normalize the spectra of the different kinds of masses and



Flow diagram.

to print out directly the normalized spectra together with the mean energies. Since the kinetic energies are as readily available in the computer as the total energies are, one may express the spectra as functions of either variable. (See discussion under *g*)).

2. Σ_i , D , d are quantities used during the calculation and the error estimate. (See below).

c) Between (3) and (4)

Here the random numbers are produced. In the present flow diagram it has been assumed that there is a subroutine available which produces (pseudo-) random numbers of three digits between 0 and 999. They are called T' and are mapped on the interval $0 \leq T_i \leq T$.

d) Between (4) and (8)

Here the $n-1$ random numbers $0 \leq T_i \leq T$ are reordered according to magnitude and the variables which determine ψ (I, 18)) are prepared.

$T = \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \dots \varepsilon_n$ is an abbreviation only.

e) Between (8) and (9) (or (14) respectively):

It is checked if perhaps one p_i is larger than the sum of the other p 's. If that is the case all the calculations up to (14) are unnecessary, since according to (I, A.10) they yield zero. It is doubtful whether this check gives a gain in computing time, as the probability for «yes» is $(\frac{1}{2})^{n-1}$ (n particles). Thus it saves only in a very few cases a long, useless calculation, but on the other hand it takes itself a time proportional to n .

f) Between (9) and (12)

the function ψ is calculated.

Σ_σ is the square bracket of (I, 18).

$$\Sigma_p \equiv \sum_i \sigma_i \sqrt{\varepsilon_i^2 - m_i^2}. \quad (\text{I, 18}).$$

That part, which begins at (11) and feeds itself back into (10) calculates the sum Σ_σ following exactly the example given in (I, Appendix). It is meant rather symbolically, since most computers do not have an operation which checks whether a number is even. There are many ways of replacing this by an operation which is much faster and implies essentially the same.

g) Between (12) and (15):

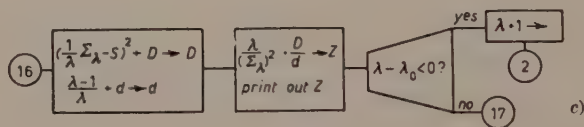
The spectra are calculated for each particle separately according to (I). W_{kr} is a storage matrix, which contains (for r fixed) the spectrum of particle with number r . See discussion under *b*).

h) Between (15) and the end.

The accuracy is checked. S contains now the sum of the ψ 's coming from N_0 samples and Σ_λ is the sum of λ such S . It seems desirable that the machine prints out S_λ and $(1/\lambda) \Sigma_\lambda$ which allows one to draw histograms such as shown in Fig. 1. This visual control often gives a much better insight than a r.m.s. value or similar quantities. Since the convergence of the MC approach seems to depend on the energy, number of particles, etc., it is advisable to have this extra check here. In order to have a summary of the actual situation, also z_λ (or z_λ^2 respectively) (see here (22) and (23)) are printed out. D and d are numerator and denominator respectively in (22) and (23) of this paper.

Finally, one may have chosen the desired accuracy so high, i.e. z_0 so small, that it would never be reached in a reasonable computing time. Therefore the possibility is provided for finishing the calculation and printing out the present results, even if z_λ is not $\leq z_0$.

If the square of the relative error is estimated by using the squared differences to the mean value, the flow diagram is changed only between (16) — ... — (17) as follows:



Flow diagram.

Since here the quadratic error is estimated, z_0 has here to be the square of the z_0 in the other method.

In the other method $\sqrt{(\lambda - 1)/\lambda}$ and $\sqrt{\pi\lambda/2}$ can be calculated for the interesting λ -values and stored as a small table. Then the lengthy square roots are eliminated from the program.

3. - Practical experiences.

At present a programme is running for nucleon nucleon collisions at 25 GeV primary (laboratory) energy. The cases $2N + \pi$ and $2N + 2\pi$ could be checked with the formulae given by BLOCK⁽⁴⁾ and showed good agreement. As the phase space integral can be solved rigorously for massless particles, we used

⁽⁴⁾ M. M. BLOCK: *Phys. Rev.*, **101**, 796 (1956).

a few such cases for various other checks. All cases in which a check was possible showed that the actual error was of the order of the predicted one and in fact in most cases it was even smaller (compare the predictions in the text between Eq. (6) and (7)).

We shall now give one typical example in some detail and summarize the others in a table.

Example: We took the following data:

$n = 3 ; \quad m_1 = m_2 = m_3 = 0 ; \quad E = T = 0.9916 \text{ } ^{(5)} ; \quad N_0 = 10 ; \quad \lambda = 37 .$

The total number of samples was therefore 370. The prescribed accuracy z_0 was taken to be zero, so that the machine continued to calculate, until it was stopped. It seems most convenient to represent everything in the form of histograms as in Fig. 1 and 2.

Fig. 2 shows the contributions S'_μ , each coming from $N_0=10$ samples. We have drawn in the mean value $\bar{S}^{(2)} = (1/\lambda)\Sigma_\lambda$ ($\lambda=37$) and the estimated error limits as well as the exact value, which in this case lies inside the estimated limits.

Fig. 3 shows $(1/\mu)\Sigma_\mu$ for $\mu = 2, 3, \dots, 37$ again together with the exact value. Here the MC value approaches the exact one from above. In other cases it came from below and in still others it oscillated.

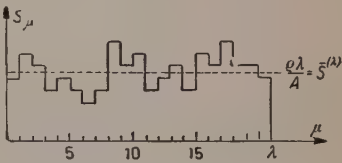


Fig. 1.

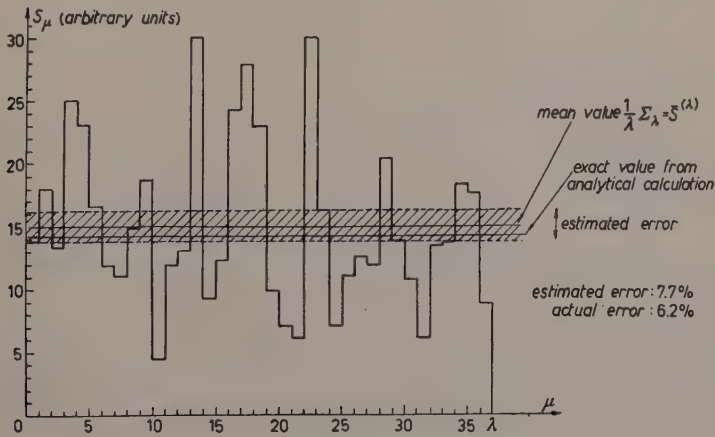


Fig. 2.

(5) Units are $\hbar = c = 1$; the energy unit is the nucleonic mass: $M = 1$.

Fig. 4 shows the estimated error in percent. One sees here that it takes some time before this *a posteriori* error estimate begins to have a meaning. In the beginning it fluctuates very much and only for λ greater than about

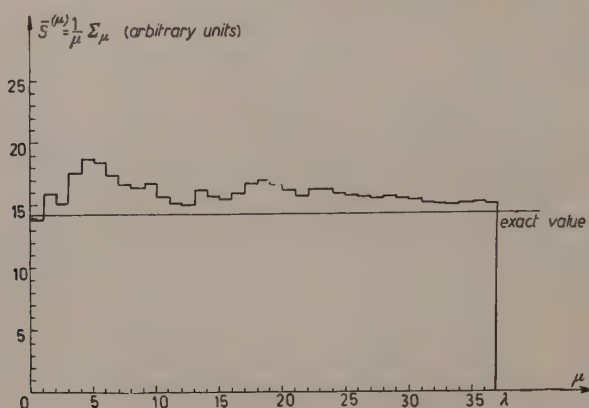


Fig. 3.

15 does it become reasonably smooth. This shows that in this case $\lambda_0 = 15$ would have been reasonable (λ_0 is that number of sets of N_0 samples, beyond which the machine would take the error estimate seriously, compare it with

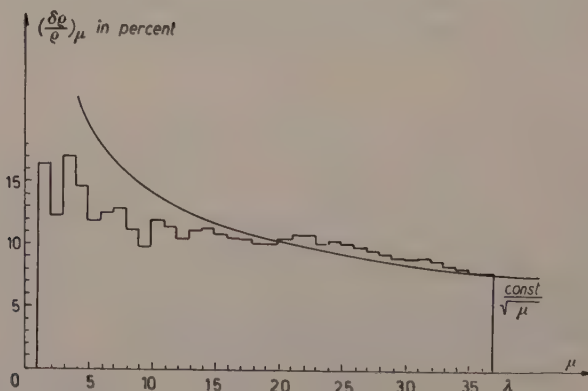


Fig. 4.

the prescribed error z_0 and stop as soon as the estimate is less than that). Of course, λ_0 is roughly inversely proportional to N_0 and here we took N_0 small on purpose, so as to obtain an illustrative example. For comparison a curve $\text{const}/\sqrt{\mu}$ has been drawn in. The constant was adjusted such that this curve passed through $(\delta \varrho / \varrho)_{(37)}$.

Fig. 5 shows the MC spectrum as a histogram together with the exact analytical expression (full curve), which was normalized to the same area.

It is interesting to compare this with not only the histogram, but with the full information, which one could obtain from it: the dotted curve shows a fit by a polynomial, the coefficients of which were calculated by a least squares method. In this calculation the normalization was accounted for by a supplementary condition. It is this dotted curve which one would consider as the calculated spectrum and it seems to come out quite well. It should be noted that it would come out even better

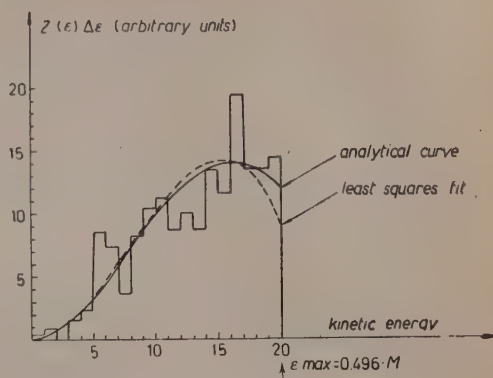


Fig. 5.

if the spectrum were smooth at the upper end. As one sees from the analytical curve, it has there the character of a step function, which makes it a bit difficult for the interpolation polynomial to reproduce it very well. For particles with mass and for higher particle numbers nothing like that will happen.

Further results for checking the method: Before application, the method was checked in a few further cases for which exact values were known. The calculations were done with only a few samples in the first two examples. The results are given in Table I.

TABLE I.

Particles	CM kinetic energy	ϱ_{MC}^*	ϱ_{exact}^*	Error	
				predicted	actual
6 particles $m = 0$	0.9916	$1.29 \cdot 10^{-6}$	$1.65 \cdot 10^{-6}$	$\pm 23 \%$	$+ 27.9\%$
6 particles $m = 0$	5.57	$5.1 \cdot 10^4$	$5.4 \cdot 10^4$	$\pm 23.5\%$	$+ 5.9\%$
3 particles $m_1 = m_2 = 1$ $m_3 = 0.15$	0.9916	15.72	14.67	$\pm 7.6\%$	$- 7.2\%$

4. - Estimates of computing time.

As already mentioned, the calculations were performed on a IBM 650 computer. Our computing times could be translated into those for other computers by a suitable factor which may be inferred from well known data on such machines.

Fig. 6 shows the computing time for 50 samples as a function of the total number of particles, n . The dotted part of the curve is an extrapolation based on estimates. It is seen that this function increases rapidly. It should be noted that:

- i) No attempt has been made in our case to achieve «optimal programming»,
- ii) there are computers available, which are much faster than the IBM 650.

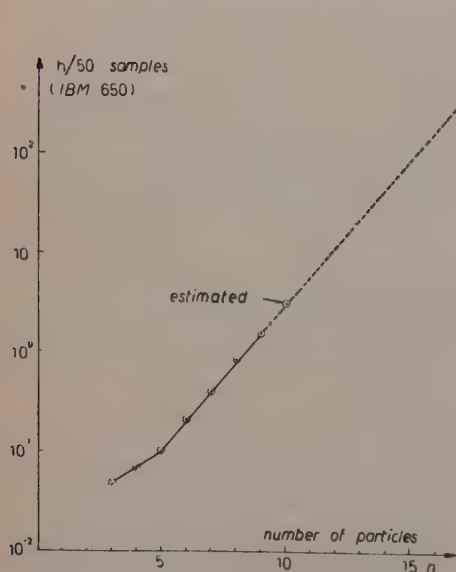


Fig. 6.

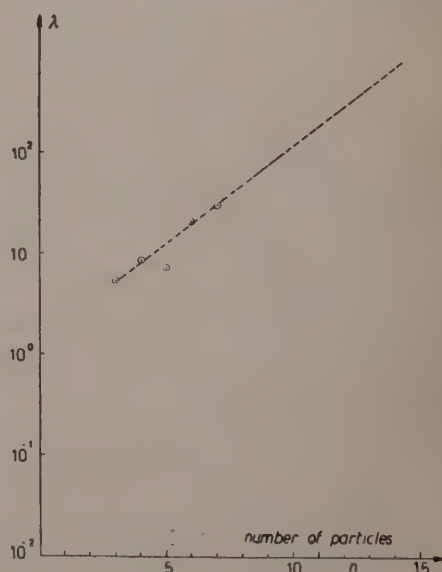


Fig. 7.

Both facts together lead to the guess that on the fastest computers this time can be reduced by a factor of about 100.

We have relatively less experience of the number of samples needed for a certain accuracy, since only a few cases were carried through until a high accuracy was achieved. Fig. 4, however, shows that the accuracy goes roughly with $1/\sqrt{\lambda}$.

In the following Fig. 7 we show the number of samples versus the number n of particles which gives an accuracy below $\sim 8\%$. Some of the points were

found by extrapolating the trend of the error by means of $\text{const}/\sqrt{\lambda}$ as shown in Fig. 4. The points partly represent mean values over several different calculations. The straight line is drawn tentatively. The present information does not allow a better determination. (For $n=5$ the error has obviously been underestimated).

It seems that in cases where the kinetic energy T is small compared to the total mass, the convergence does not depend very much on the number of particles. In contrast if T is larger than M , the number of necessary samples increases rapidly with the number of particles. This can be made plausible by a discussion of the formulae for ψ (I, 18). Taking two different samples and looking at what effect the differences of the samples have on ψ , it is found that ψ is more sensitive for differences of samples when $T \gg M$ than when the mass is large, since in the latter case the masses determine the value of ψ more than the kinetic energies do.

Sensitivity with respect to differences between samples means slow convergence. It seems that Fig. 6 shows a pessimistic estimate, since it is drawn using mostly results with cases where T was larger than M .

The total computing time necessary for reaching an accuracy of about 8% is then given by the product of the curves of Figs. 6 and 7. The result is shown in Fig. 8.

This figure should not be taken too seriously because:

- i) it is based partly on very rough estimates,
- ii) in most applications for higher numbers of created particles less accuracy is necessary, since the total phase space volume is already so small that it contributes very little to everything which one normally calculates, as for instance the mean number of created particles, total spectra, mean kinetic energies, etc.

Assuming that optimal programming on a faster computer leads to perhaps a reduction by a factor 100, then the method is expected to break down for 13 particles if 8% accuracy is demanded throughout. If, however, the higher particle numbers are not very important, so that (20 ÷ 30)% accuracy is enough,

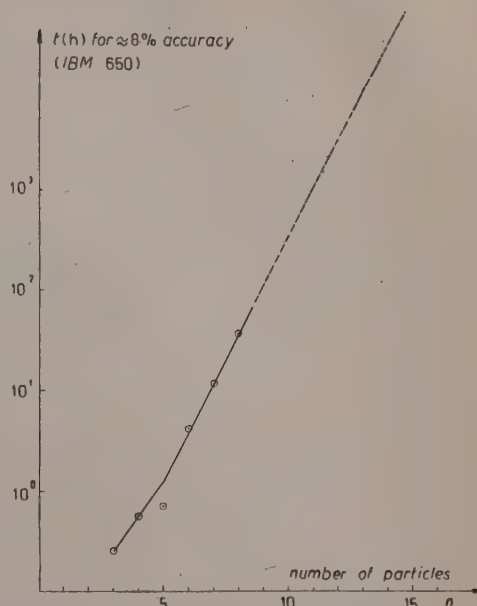


Fig. 8.

then one may use the method up to 14 or perhaps 15 particles. At energies where *e.g.* in a nucleon-nucleon collision 13 mesons are produced with non-vanishing probability, the whole Fermi theory becomes rather doubtful on one hand and pure thermodynamical approaches begin to yield reasonable results on the other hand. Thus the MC method seems to cover essentially the most interesting region.

5. - Comparison with the usual approximation.

The usual procedure for calculating a phase space integral for, say, N nucleons and n mesons has been to consider the nucleons as very heavy and the mesons as particles without mass. Then one puts

$$(24) \quad E = \sqrt{p^2 + m^2} \approx \begin{cases} p & \text{for mesons} \\ m + \frac{p^2}{2m} & \text{for nucleons} \end{cases}$$

and one can now calculate exactly on either assumption the integral

$$(25) \quad \varrho_k^*(E, P) = \int \delta(E - \sum_i E_i) \delta(P - \sum_i P_i) dP_1 \dots dP$$

by introducing the Fourier representations of the two δ -functions. The total ϱ^* is then given by the convolution

$$(26) \quad \varrho_{n,N}^*(E, P) = \int \varrho_n^*(E - \varepsilon, |P - p|) \varrho_N^*(\varepsilon, p) d\varepsilon dp,$$

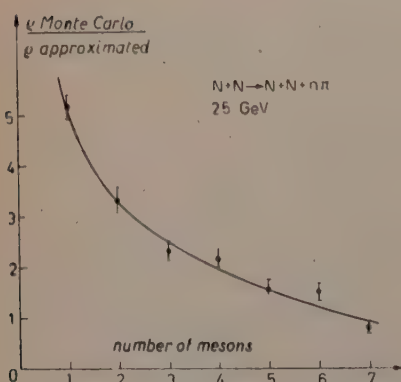


Fig. 9.

which in turn is quite involved but can be calculated by various approximations, *e.g.* the method of steepest descent or the use of tabulated functions. As far as we know, only BELEN'KIJ *et al.* ⁽⁶⁾ have considered the convolution (26), whereas other authors tacitly assume that each system of mesons and nucleons, separately has total momentum zero. This would be a fair approximation only for the case that any of these systems contains many particles. Since this is not true, the result of such calculations will be even worse than with

⁽⁶⁾ S. BELEN'KIJ *et al.*: *Usp. Fiz. Nauk*, **62**, 1 (1957).

the use of (26). As an illustration, Fig. 9 shows the factor between ϱ^* calculated using (24)–(26) and ϱ^* calculated by our method. We took 25 GeV primary energy.

* * *

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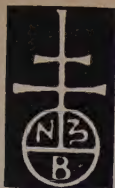
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